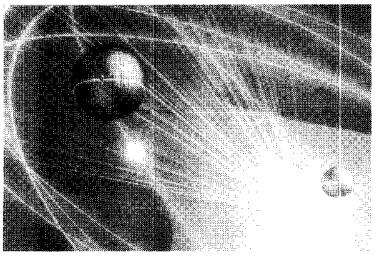

CHAPTER 2

Differentiation

Introduction Two fundamental problems are considered in calculus. The **problem of slopes** is concerned with finding the slope of (the tangent line to) a given curve at a given point on the curve. The **problem of areas** is concerned with finding the area of a plane region bounded by curves and straight lines. The solution of the problem of slopes is the subject of **differential calculus**. As we will see, it has many applications in mathematics and other disciplines. The problem of areas is the subject of **integral calculus**, which we begin in Chapter 5.

2.1 Tangent Lines and Their Slopes

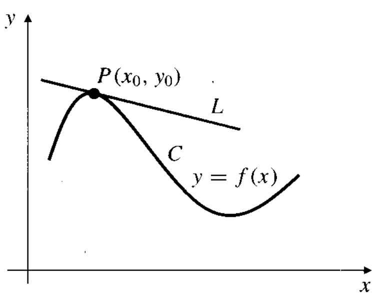

Figure 2.1 L is tangent to C at P

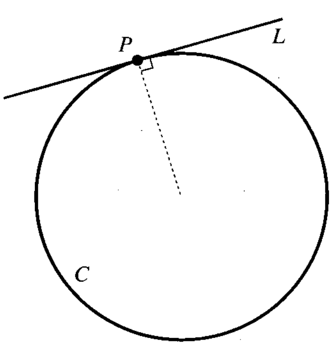

Figure 2.2 L is tangent to C at P

This section deals with the problem of finding a straight line L that is tangent to a curve C at a point P. As is often the case in mathematics, the most important step in the solution of such a fundamental problem is making a suitable definition.

For simplicity, and to avoid certain problems best postponed until later, we will not deal with the most general kinds of curves now, but only with those that are the *graphs of continuous functions*. Let C be the graph of $y = f(x)$ and let P be the point (x_0, y_0) on C, so that $y_0 = f(x_0)$. We assume that P is not an endpoint of C. Therefore, C extends some distance on both sides of P. (See Figure 2.1.)

What do we mean when we say that the line L is tangent to C at P? Past experience with tangent lines to circles does not help us to define tangency for more general curves. A tangent line to a circle (Figure 2.2) has the following properties:

(i) It meets the circle at only one point.

(ii) The circle lies on only one side of the line.

(iii) The tangent is perpendicular to the line joining the centre of the circle to the point of contact.

Most curves do not have obvious *centres*, so (iii) is useless for characterizing tangents to them. The curves in Figure 2.3 show that (i) and (ii) cannot be used to define tangency either. In particular, Figure 2.3(d) is not "smooth" at P so that curve should not have any tangent line there. A tangent line should have the "same direction" as the curve does at the point of tangency.

A reasonable definition of tangency can be stated in terms of limits. If Q is a point on C different from P, then the line through P and Q is called a **secant line** to the curve. This line rotates around P as Q moves along the curve. If L is a line through P whose slope is the limit of the slopes of these secant lines PQ as Q approaches P along C (Figure 2.4), then L is tangent to C at P.

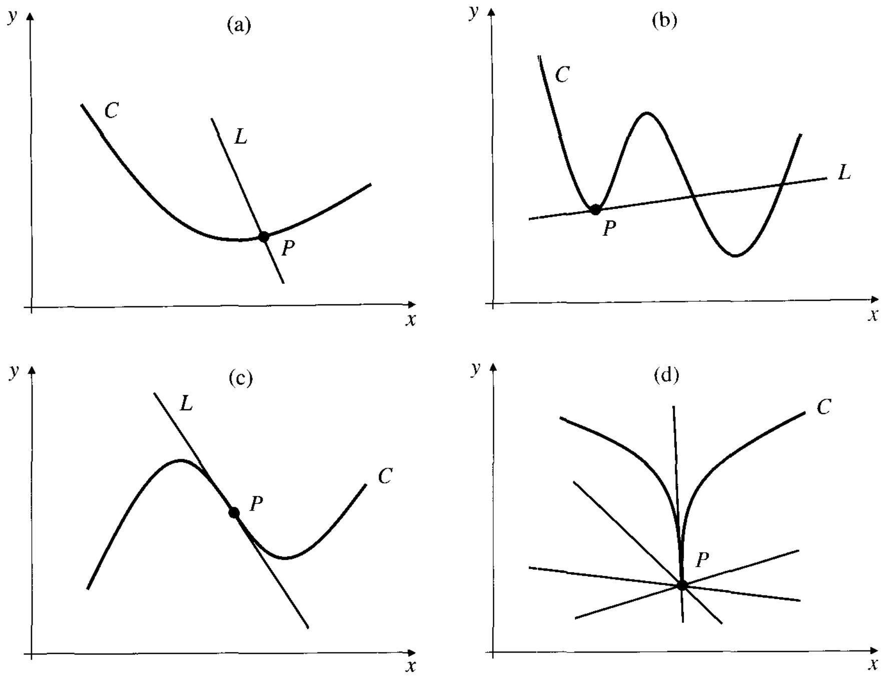

Figure 2.3

(a) L meets C only at P but is not tangent to C

(b) L meets C at several points but is tangent to C at P

(c) L is tangent to C at P but crosses C at P

(d) Many lines meet C only at P but none of them is tangent to C at P

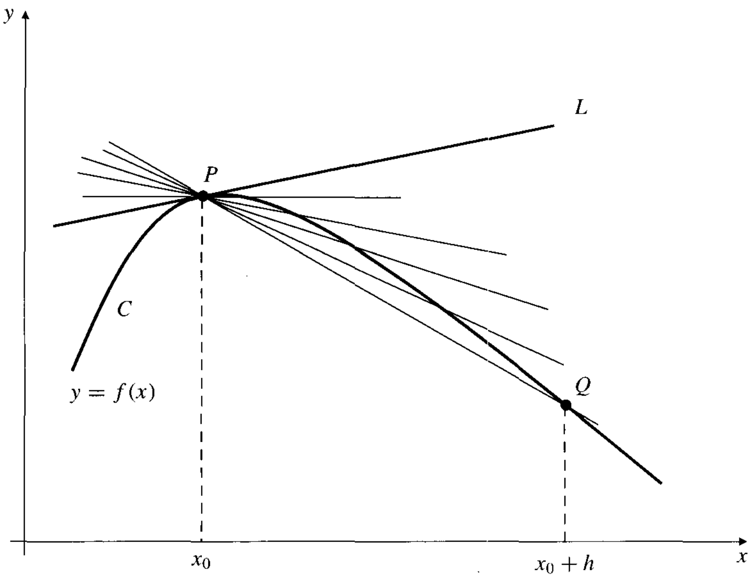

Figure 2.4 Secant lines PQ approach tangent line L as Q approaches P along the curve C

Since C is the graph of the *function* $y = f(x)$, then vertical lines can meet C only once. Since $P = (x_0, f(x_0))$, a different point Q on the graph must have a different x-coordinate, say $x_0 + h$, where $h \neq 0$. Thus $Q = (x_0 + h, f(x_0 + h))$, and the slope of the line PQ is

$$\frac{f(x_0 + h) - f(x_0)}{h}.$$

This expression is called the **Newton quotient** or **difference quotient** for f at x_0. Note that h can be positive or negative, depending on whether Q is to the right or left of P.

DEFINITION 1

Nonvertical tangent lines

Suppose that the function f is continuous at $x = x_0$ and that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = m$$

exists. Then the straight line having slope m and passing through the point $P = (x_0, f(x_0))$ is called the **tangent line** (or simply the **tangent**) to the graph of $y = f(x)$ at P. An equation of this tangent is

$$y = m(x - x_0) + y_0.$$

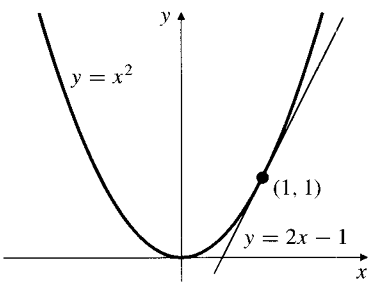

Figure 2.5 The tangent to $y = x^2$ at $(1, 1)$

Example 1 Find an equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

Solution Here $f(x) = x^2$, $x_0 = 1$, and $y_0 = f(1) = 1$. The slope of the required tangent is:

$$\begin{aligned} m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} 2 + h = 2. \end{aligned}$$

Accordingly, the equation of the tangent line at $(1, 1)$ is $y = 2(x - 1) + 1$, or $y = 2x - 1$. See Figure 2.5.

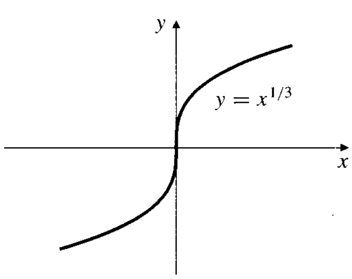

Figure 2.6 The y-axis is tangent to $y = x^{1/3}$ at the origin

Definition 1 deals only with tangents that have finite slopes and are, therefore, not vertical. It is also possible for the graph of a continuous function to have a *vertical* tangent line.

Example 2 Consider the graph of the function $f(x) = \sqrt[3]{x} = x^{1/3}$, which is shown in Figure 2.6. The graph is a smooth curve, and it seems evident that the y-axis is tangent to this curve at the origin. Let us try to calculate the limit of the Newton quotient for f at $x = 0$:

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty.$$

Although the limit does not exist, the slope of the secant line joining the origin to another point Q on the curve approaches infinity as Q approaches the origin from either side.

Example 3 On the other hand, the function $f(x) = x^{2/3}$, whose graph is shown in Figure 2.7, does not have a tangent line at the origin because it is not "smooth" there. In this case the Newton quotient is

$$\frac{f(0+h) - f(0)}{h} = \frac{h^{2/3}}{h} = \frac{1}{h^{1/3}},$$

which has no limit as h approaches zero. (The right limit is ∞; the left limit is $-\infty$.) We say this curve has a **cusp** at the origin. A cusp is an infinitely sharp point; if you were travelling along the curve, you would have to stop and turn 180° at the origin. ■

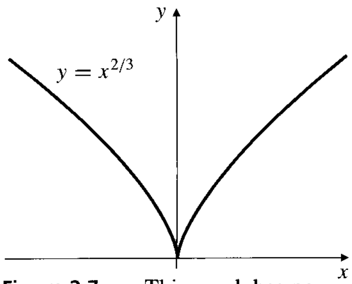

Figure 2.7 This graph has no tangent at the origin

In the light of the two examples above, we extend the definition of tangent line to allow for vertical tangents as follows:

DEFINITION 2

Vertical tangents

If f is continuous at $P = (x_0, y_0)$, where $y_0 = f(x_0)$, and if either

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \infty \quad \text{or} \quad \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = -\infty,$$

then the vertical line $x = x_0$ is tangent to the graph $y = f(x)$ at P. If the limit of the Newton quotient fails to exist in any other way than by being ∞ or $-\infty$, the graph $y = f(x)$ has no tangent line at P.

Example 4 Does the graph of $y = |x|$ have a tangent line at $x = 0$?

Solution The Newton quotient here is

$$\frac{|0+h| - |0|}{h} = \frac{|h|}{h} = \operatorname{sgn} h = \begin{cases} 1, & \text{if } h > 0 \\ -1, & \text{if } h < 0. \end{cases}$$

Since $\operatorname{sgn} h$ has different right and left limits at 0 (namely, 1 and -1), the Newton quotient has no limit as $h \to 0$, so $y = |x|$ has no tangent line at $(0, 0)$. (See Figure 2.8.) The graph does not have a cusp at the origin, but it is kinked at that point; *it suddenly changes direction and is not smooth.* Curves have tangents only at points where they are smooth. The graphs of $y = x^{2/3}$ and $y = |x|$ have tangent lines everywhere except at the origin, where they are not smooth. ■

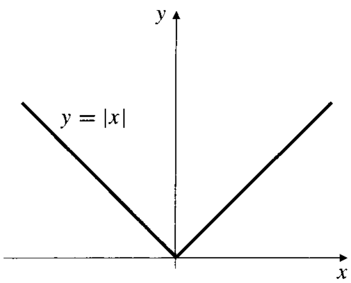

Figure 2.8 $y = |x|$ has no tangent at the origin

DEFINITION 3

The slope of a curve

The **slope** of a curve C at a point P is the slope of the tangent line to C at P if such a tangent line exists. In particular, the slope of the graph of $y = f(x)$ at the point x_0 is

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Example 5 Find the slope of the curve $y = x/(3x+2)$ at the point $x = -2$.

Solution If $x = -2$, then $y = 1/2$, so the required slope is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{\dfrac{-2+h}{3(-2+h)+2} - \dfrac{1}{2}}{h} \\ &= \lim_{h\to 0} \frac{-4+2h-(-6+3h+2)}{2(-6+3h+2)h} \\ &= \lim_{h\to 0} \frac{-h}{2h(-4+3h)} = \lim_{h\to 0} \frac{-1}{2(-4+3h)} = \frac{1}{8}. \end{aligned}$$

■

Normals

If a curve C has a tangent line L at point P, then the straight line N through P perpendicular to L is called the **normal** to C at P. If L is horizontal, then N is vertical; if L is vertical, then N is horizontal. If L is neither horizontal nor vertical, then, as shown in Section P.2, the slope of N is the negative reciprocal of the slope of L:

$$\text{slope of the normal} = \frac{-1}{\text{slope of the tangent}}.$$

Example 6 Find an equation of the normal to $y = x^2$ at $(1, 1)$.

Solution By Example 1, the tangent to $y = x^2$ at $(1, 1)$ has slope 2. Hence the normal has slope $-1/2$, and its equation is

$$y = -\frac{1}{2}(x-1)+1 \qquad \text{or} \qquad y = -\frac{x}{2}+\frac{3}{2}.$$

■

Example 7 Find equations of the straight lines that are tangent and normal to the curve $y = \sqrt{x}$ at the point $(4, 2)$.

Solution The slope of the tangent at $(4, 2)$ (Figure 2.9) is

$$\begin{aligned} m = \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} &= \lim_{h\to 0} \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} \\ &= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}. \end{aligned}$$

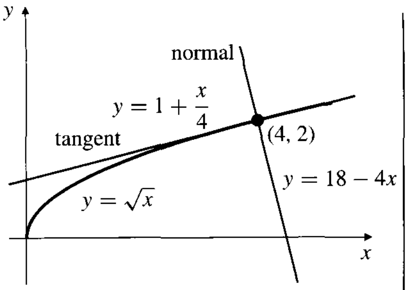

Figure 2.9 The tangent and normal to $y = \sqrt{x}$ at (4, 2)

The tangent line has equation

$$y = \frac{1}{4}(x - 4) + 2 \qquad \text{or} \qquad x - 4y + 4 = 0,$$

and the normal has equation

$$y = -4(x - 4) + 2 \qquad \text{or} \qquad y = -4x + 18.$$

■

Exercises 2.1

In Exercises 1–12, find an equation of the straight line tangent to the given curve at the point indicated.

1. $y = 3x - 1$ at $(1, 2)$
2. $y = x/2$ at $(a, a/2)$
3. $y = 2x^2 - 5$ at $(2, 3)$
4. $y = 6 - x - x^2$ at $x = -2$
5. $y = x^3 + 8$ at $x = -2$
6. $y = \dfrac{1}{x^2 + 1}$ at $(0, 1)$
7. $y = \sqrt{x + 1}$ at $x = 3$
8. $y = \dfrac{1}{\sqrt{x}}$ at $x = 9$
9. $y = \dfrac{2x}{x + 2}$ at $x = 2$
10. $y = \sqrt{5 - x^2}$ at $x = 1$
11. $y = x^2$ at $x = x_0$
12. $y = \dfrac{1}{x}$ at $\left(a, \dfrac{1}{a}\right)$

Do the graphs of the functions f in Exercises 13–17 have tangent lines at the given points? If yes, what is the tangent line?

13. $f(x) = \sqrt{|x|}$ at $x = 0$
14. $f(x) = (x - 1)^{4/3}$ at $x = 1$
15. $f(x) = (x + 2)^{3/5}$ at $x = -2$
16. $f(x) = |x^2 - 1|$ at $x = 1$
17. $f(x) = \begin{cases} \sqrt{x} & \text{if } x \ge 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$ at $x = 0$
18. Find the slope of the curve $y = x^2 - 1$ at the point $x = x_0$. What is the equation of the tangent line to $y = x^2 - 1$ that has slope -3?
19. (a) Find the slope of $y = x^3$ at the point $x = a$.
    (b) Find the equations of the straight lines having slope 3 that are tangent to $y = x^3$.
20. Find all points on the curve $y = x^3 - 3x$ where the tangent line is parallel to the x-axis.
21. Find all points on the curve $y = x^3 - x + 1$ where the tangent line is parallel to the line $y = 2x + 5$.
22. Find all points on the curve $y = 1/x$ where the tangent line is perpendicular to the line $y = 4x - 3$.
23. For what value of the constant k is the line $x + y = k$ normal to the curve $y = x^2$?
24. For what value of the constant k do the curves $y = kx^2$ and $y = k(x - 2)^2$ intersect at right angles. *Hint:* where do the curves intersect? What are their slopes there?

Use a graphics utility to plot the following curves. Where does the curve have a horizontal tangent? Does the curve fail to have a tangent line anywhere?

25. $y = x^3(5 - x)^2$
26. $y = 2x^3 - 3x^2 - 12x + 1$
27. $y = |x^2 - 1| - x$
28. $y = |x + 1| - |x - 1|$
29. $y = (x^2 - 1)^{1/3}$
30. $y = ((x^2 - 1)^2)^{1/3}$

*31. If line L is tangent to curve C at point P, then the smaller angle between L and the secant line PQ joining P to another point Q on C approaches 0 as Q approaches P along C. Is the converse true: if the angle between PQ and line L (which passes through P) approaches 0, must L be tangent to C?

*32. Let $P(x)$ be a polynomial. If a is a real number, then $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n$$

for some $n \ge 0$. If $\ell(x) = m(x - a) + b$, show that the straight line $y = \ell(x)$ is tangent to the graph of $y = P(x)$ at $x = a$ provided $P(x) - \ell(x) = (x - a)^2 Q(x)$, where $Q(x)$ is a polynomial.

2.2 The Derivative

A straight line has the property that its slope is the same at all points. For any other graph, however, the slope may vary from point to point. Thus the slope of the graph of $y = f(x)$ at the point x is itself a function of x. At any point x where the graph has a finite slope we say that f is differentiable, and we call the slope the derivative of f. The derivative is therefore the limit of the Newton quotient.

DEFINITION 4

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.

The domain of the derivative f' (read "f prime") is the set of numbers x in the domain of f where the graph of f has a *nonvertical* tangent line, and the value $f'(x_0)$ of f' at such a point x_0 is the slope of the tangent line to $y = f(x)$ there. Thus the equation of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

The domain $\mathcal{D}(f')$ of f' may be smaller than the domain $\mathcal{D}(f)$ of f because it contains only those points in $\mathcal{D}(f)$ at which f is differentiable. Values of x in $\mathcal{D}(f)$ where f is not differentiable are called **singular points** of f.

Remark The value of the derivative of f at a particular point x_0 can be expressed as a limit in either of two ways:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

In the second limit $x_0 + h$ is replaced by x, so that $h = x - x_0$ and $h \to 0$ is equivalent to $x \to x_0$.

The process of calculating the derivative f' of a given function f is called **differentiation**. The graph of f' can often be sketched directly from that of f by visualizing slopes, a procedure called **graphical differentiation**. In Figure 2.10 the graphs of f' and g' were obtained by measuring the slopes at the corresponding points in the graphs of f and g lying above them. The height of the graph $y = f'(x)$ at x is the slope of the graph of $y = f(x)$ at x. Note that -1 and 1 are singular points of f. $f(-1)$ and $f(1)$ are defined, but $f'(-1)$ and $f'(1)$ are not defined; the graph of f has no tangent at -1 or at 1.

A function is differentiable on a set S if it is differentiable at every point x in S. Typically, the functions we encounter are defined on intervals or unions of intervals. If f is defined on a closed interval $[a, b]$, Definition 4 does not allow for the existence of a derivative at the endpoints $x = a$ or $x = b$. (Why?) As

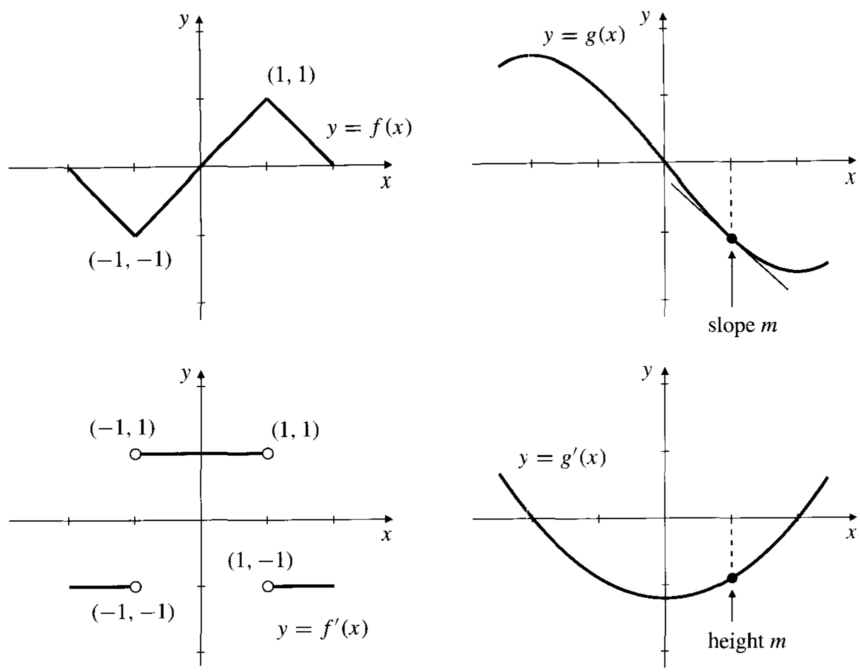

Figure 2.10 Graphical differentiation

we did for continuity in Section 1.4, we extend the definition to allow for a **right derivative** at $x = a$ and a **left derivative** at $x = b$:

$$f'_+(a) = \lim_{h\to 0+} \frac{f(a+h) - f(a)}{h}, \qquad f'_-(b) = \lim_{h\to 0-} \frac{f(b+h) - f(b)}{h}.$$

We now say that f is **differentiable** on $[a, b]$ if $f'(x)$ exists for all x in $]a, b[$ and $f'_+(a)$ and $f'_-(b)$ both exist.

Some Important Derivatives

We now give several examples of the calculation of derivatives algebraically from the definition of derivative. Some of these are the basic building blocks from which more complicated derivatives can be calculated later. They are collected in Table 1 later in this section and should be memorized.

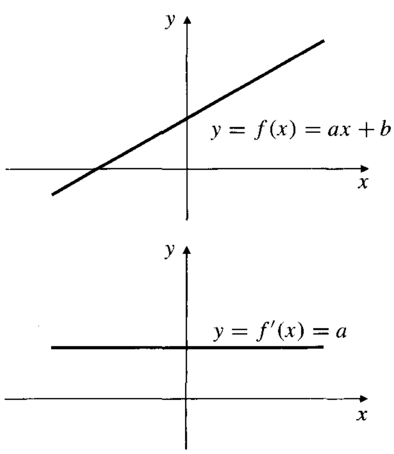

Figure 2.11 The derivative of the linear function $f(x) = ax + b$ is the constant function $f'(x) = a$

Example 1 **(The derivative of a linear function)** Show that if $f(x) = ax + b$, then $f'(x) = a$.

Solution The result is apparent from the graph of f (Figure 2.11), but we will do the calculation using the definition:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a(x+h) + b - (ax+b)}{h} \\ &= \lim_{h\to 0} \frac{ah}{h} = a. \end{aligned}$$

■

An important special case of Example 1 says that the derivative of a constant function is the zero function:

If $g(x) = c$ (constant), then $g'(x) = 0$.

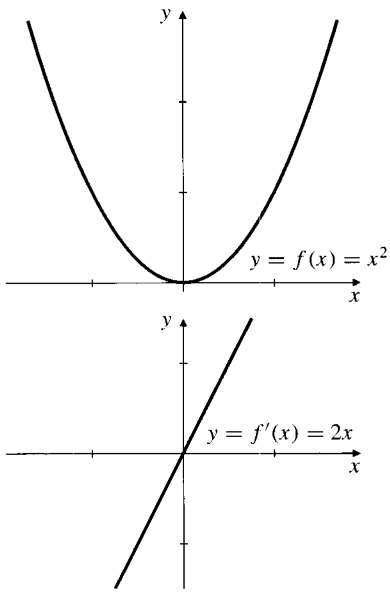

Figure 2.12 The derivative of $f(x) = x^2$ is $f'(x) = 2x$

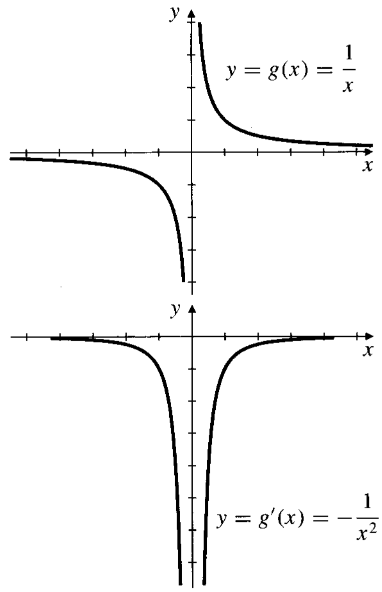

Figure 2.13 The derivative of $g(x) = 1/x$ is $g'(x) = -1/x^2$

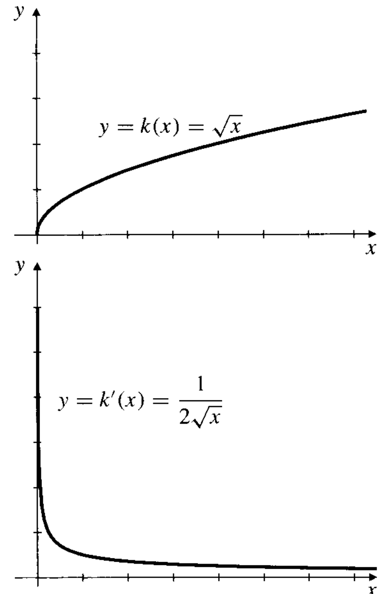

Figure 2.14 The derivative of $k(x) = \sqrt{x}$ is $k'(x) = 1/(2\sqrt{x})$

Example 2 Use the definition of the derivative to calculate the derivatives of the functions:

$$\text{(a) } f(x) = x^2, \quad \text{(b) } g(x) = \frac{1}{x}, \quad \text{and} \quad \text{(c) } k(x) = \sqrt{x}.$$

Solution See Figures 2.12–2.14 for the graphs of these functions and of their derivatives.

$$\begin{aligned}
\text{(a)}\quad f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0}(2x + h) = 2x.
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h(x+h)x} = \lim_{h\to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2}.
\end{aligned}$$

(c)
$$
\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}
$$

Note that k is not differentiable at $x = 0$. (See Figure 2.14.) Since 0 is in the domain of k, it is a singular point of k.

The three derivative formulas calculated in Example 2 are special cases of the following **General Power Rule**:

If $f(x) = x^r$, then $f'(x) = r\,x^{r-1}$.

This formula, which we will verify in Section 3.3, is valid for *all values of r and x for which x^{r-1} makes sense as a real number.*

Example 3 **(Differentiating powers)**

If $f(x) = x^{5/3}$, then $f'(x) = \frac{5}{3}x^{(5/3)-1} = \frac{5}{3}x^{2/3}$ for all real x.

If $g(t) = \frac{1}{\sqrt{t}} = t^{-1/2}$, then $g'(t) = -\frac{1}{2}t^{-(1/2)-1} = -\frac{1}{2}t^{-3/2}$ for $t > 0$.

Eventually we will prove all appropriate cases of the General Power Rule. For the time being, here is a proof of the case $r = n$, a positive integer, based on the *factoring of a difference of nth powers*:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}).$$

(Check this formula by multiplying the two factors on the right-hand side.) If $f(x) = x^n$, $a = x+h$ and $b = x$, then $a - b = h$ and,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{h\,\overbrace{[(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \cdots + x^{n-1}]}^{n \text{ terms}}}{h} \\
&= nx^{n-1}.
\end{aligned}
$$

An alternative proof based on the product rule and mathematical induction will be given in Section 2.3. The factorization method used above can also be used to demonstrate the General Power Rule for negative integers, $r = -n$, and reciprocals of integers, $r = 1/n$. (See Exercises 50 and 52 at the end of this section.)

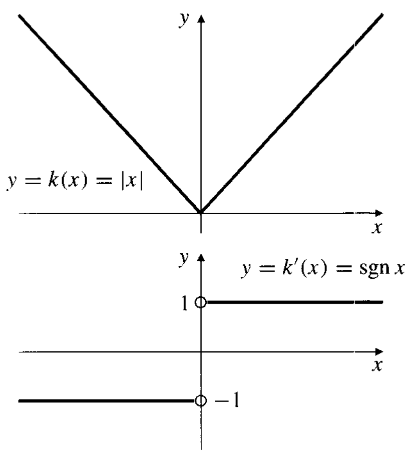

Figure 2.15 The derivative of $|x|$ is $\operatorname{sgn} x = x/|x|$

Example 4 **(Differentiating the absolute value function)** Verify that:

$$\text{If } f(x) = |x|, \quad \text{then} \quad f'(x) = \frac{x}{|x|} = \operatorname{sgn} x.$$

Solution We have

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

Thus, from Example 1 above, $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. Also, Example 4 of Section 2.1 shows that f is not differentiable at $x = 0$, which is a singular point of f. Therefore (see Figure 2.15),

$$f'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases} = \frac{x}{|x|} = \operatorname{sgn} x.$$

■

Table 1 lists a collection of the elementary derivatives calculated above. Beginning in Section 2.3 we will develop general rules for calculating the derivatives of functions obtained by combining simpler functions. Thereafter we will seldom have to revert to the definition of the derivative and to the calculation of limits to evaluate derivatives. It is important, therefore, to remember the derivatives of some elementary functions. Memorize those in Table 1.

Table 1. Some elementary functions and their derivatives

$f(x)$	$f'(x)$
c (constant)	0
x	1
x^2	$2x$
$\frac{1}{x}$	$-\frac{1}{x^2}$ $(x \neq 0)$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$ $(x > 0)$
x^r	$r\,x^{r-1}$ (x^{r-1} real)
$\lvert x\rvert$	$\frac{x}{\lvert x\rvert} = \operatorname{sgn} x$

Leibniz Notation

Because functions can be written in different ways, it is useful to have more than one notation for derivatives. If $y = f(x)$, we can use the dependent variable y to represent the function, and we can denote the derivative of the function with respect to x in any of the following ways:

$$D_x y = y' = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = Df(x).$$

Often the most convenient way of referring to the derivative of a function given explicitly as an expression in the variable x is to write $\frac{d}{dx}$ in front of that expression. The symbol $\frac{d}{dx}$ is a *differential operator* and should be read "the derivative with respect to x of ..." For example,

$$\frac{d}{dx}x^2 = 2x \quad \text{(the derivative with respect to } x \text{ of } x^2 \text{ is } 2x\text{)}$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dt}t^{100} = 100t^{99}$$

$$\text{if } y = u^3, \text{ then } \frac{dy}{du} = 3u^2.$$

The value of the derivative of a function at a particular number x_0 in its domain can also be expressed in several ways:

$$D_x y\Big|_{x=x_0} = y'\Big|_{x=x_0} = \frac{dy}{dx}\Big|_{x=x_0} = \frac{d}{dx}f(x)\Big|_{x=x_0} = f'(x_0) = Df(x_0).$$

Do not confuse the expressions

$$\frac{d}{dx}f(x) \text{ and } \frac{d}{dx}f(x)\Big|_{x=x_0}.$$

The first expression represents a *function*, $f'(x)$. The second represents a *number*, $f'(x_0)$.

The symbol $\Big|_{x=x_0}$ is called an **evaluation symbol**. It signifies that the expression preceding it should be evaluated at $x = x_0$. Thus,

$$\frac{d}{dx}x^4\Big|_{x=-1} = 4x^3\Big|_{x=-1} = 4(-1)^3 = -4.$$

Here is another example in which a derivative is computed from the definition, this time for a somewhat more complicated function.

Example 5 Use the definition of derivative to calculate $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)\Big|_{x=2}$.

Solution We could calculate $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$ and then substitute $x = 2$, but it is easier to put $x = 2$ in the expression for the Newton quotient before taking the limit:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{x^2+1}\right)\Big|_{x=2} &= \lim_{h\to 0}\frac{\dfrac{2+h}{(2+h)^2+1} - \dfrac{2}{2^2+1}}{h}\\ &= \lim_{h\to 0}\frac{\dfrac{2+h}{5+4h+h^2} - \dfrac{2}{5}}{h}\\ &= \lim_{h\to 0}\frac{5(2+h) - 2(5+4h+h^2)}{5(5+4h+h^2)h}\\ &= \lim_{h\to 0}\frac{-3h-2h^2}{5(5+4h+h^2)h}\\ &= \lim_{h\to 0}\frac{-3-2h}{5(5+4h+h^2)} = -\frac{3}{25}.\end{aligned}$$

■

The notations dy/dx and $\frac{d}{dx}f(x)$ are called **Leibniz notations** for the derivative, after Gottfried Wilhelm Leibniz (1646–1716), one of the creators of calculus, who used such notations. The main ideas of calculus were developed independently by Leibniz and Isaac Newton (1642–1727); the latter used notations similar to the prime (y') notations we use here.

The Leibniz notation is suggested by the definition of derivative. The Newton quotient $[f(x+h) - f(x)]/h$, whose limit we take to find the derivative dy/dx, can be written in the form $\Delta y/\Delta x$, where $\Delta y = f(x+h) - f(x)$ is the increment in y, and $\Delta x = (x+h) - x = h$ is the corresponding increment in x as we pass from the point $(x, f(x))$ to the point $(x+h, f(x+h))$ on the graph of f. (See Figure 2.16.) Using symbols:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

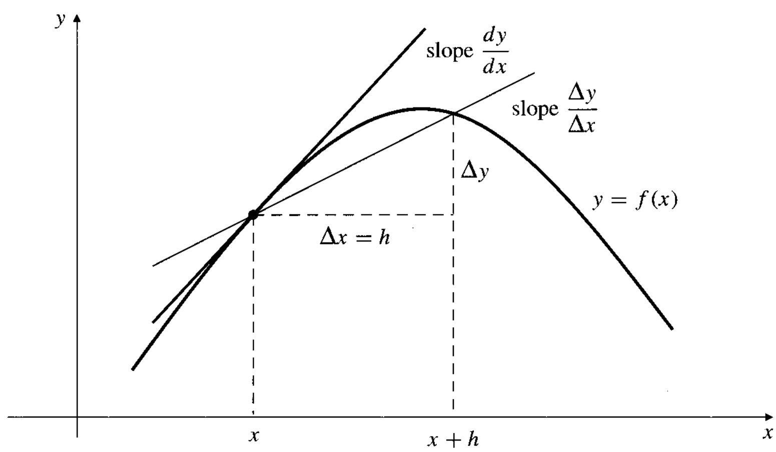

Figure 2.16 $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

Differentials

The Newton quotient $\Delta y/\Delta x$ is actually the quotient of two quantities, Δy and Δx. It is not at all clear, however, that the derivative dy/dx, the limit of $\Delta y/\Delta x$ as Δx approaches zero, can be regarded as a quotient. If y is a continuous function of x, then Δy approaches zero when Δx approaches zero, so dy/dx appears to be the meaningless quantity $0/0$. Nevertheless, it is sometimes useful to be able to refer to quantities dy and dx in such a way that their quotient is the derivative dy/dx. We can justify this by regarding dx as a new *independent* variable (called **the differential of x**) and defining a new *dependent* variable dy (**the differential of y**) as a function of x and dx by

$$dy = \frac{dy}{dx}dx = f'(x)\,dx.$$

For example, if $y = x^2$, we can write $dy = 2x\,dx$ to mean the same thing as $dy/dx = 2x$. This *differential notation* will be used for the interpretation and manipulation of integrals beginning in Chapter 5.

Note that, defined as above, differentials are merely variables that may or may not be small in absolute value. The differentials dy and dx were originally regarded (by Leibniz and his successors) as "infinitesimals" (infinitely small but nonzero) quantities whose quotient dy/dx gave the slope of the tangent line (a secant line meeting the graph of $y = f(x)$ at two points infinitely close together). It can be shown that such "infinitesimal" quantities cannot exist (as real numbers). It is possible to extend the number system to contain infinitesimals and use these to develop calculus, but we will not consider this approach here.

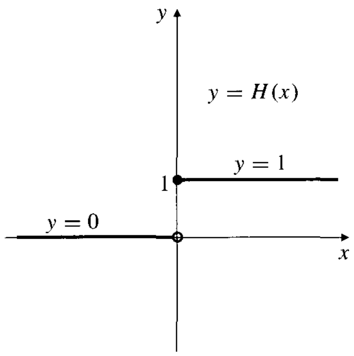

Figure 2.17 This function is not a derivative on $[-1, 1]$; it does not have the intermediate-value property.

Derivatives Have the Intermediate-Value Property

Is a function f defined on an interval I necessarily the derivative of some other function defined on I? The answer is no; some functions are derivatives and some are not. Although a derivative need not be a continuous function, it must, like a continuous function, have the intermediate-value property: on an interval $[a, b]$, a derivative $f'(x)$ takes on every value between $f'(a)$ and $f'(b)$. An everywhere-defined step function such as the Heaviside $H(x)$ function considered in Example 1 in Section 1.4 does not have this property on, say, the interval $[-1, 1]$, so cannot be the derivative of a function on that interval. This argument does not apply to the signum function, which is the derivative of the absolute value function on any interval (see Example 4), even though it does not have the intermediate-value property on an interval containing the origin. Note, however, that the signum function is not defined at the origin.

If $g(x)$ is continuous on an interval I, then $g(x) = f'(x)$ for some function f that is differentiable on I. We will discuss this further in Chapter 5 and prove it in Appendix III.

Exercises 2.2

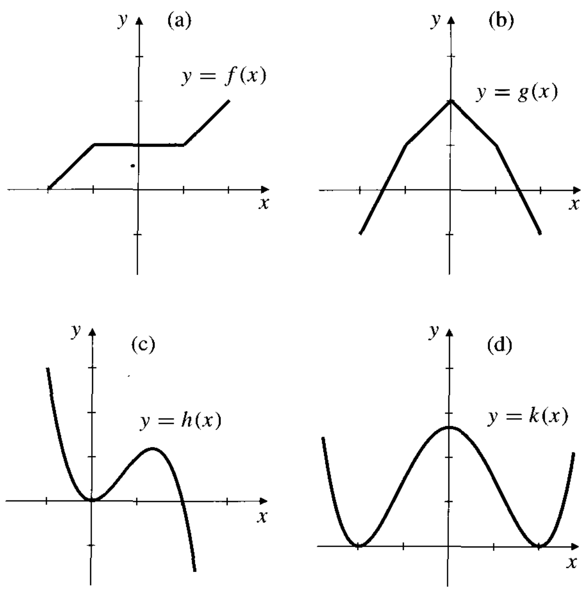

Figure 2.18

Make rough sketches of the graphs of the derivatives of the functions in Exercises 1–4.

1. The function f graphed in Figure 2.18(a).
2. The function g graphed in Figure 2.18(b).
3. The function h graphed in Figure 2.18(c).
4. The function k graphed in Figure 2.18(d).
5. Where is the function f graphed in Figure 2.18(a) differentiable?
6. Where is the function g graphed in Figure 2.18(b) differentiable?

Use a graphics utility with differentiation capabilities to plot the graphs of the following functions and their derivatives. Observe the relationships between the graph of y and that of y' in each case. What features of the graph of y can you infer from the graph of y'?

7. $y = 3x - x^2 - 1$
8. $y = x^3 - 3x^2 + 2x + 1$
9. $y = |x^3 - x|$
10. $y = |x^2 - 1| - |x^2 - 4|$

In Exercises 11–22, calculate the derivative of the given function directly from the definition of derivative.

11. $y = x^2 - 3x$
12. $f(x) = 1 + 4x - 5x^2$
13. $f(x) = x^3$
14. $s = \dfrac{1}{3 + 4t}$
15. $F(t) = \sqrt{2t + 1}$
16. $f(x) = \dfrac{3}{4}\sqrt{2 - x}$

17. $y = x + \dfrac{1}{x}$

18. $z = \dfrac{s}{1+s}$

19. $F(x) = \dfrac{1}{\sqrt{1+x^2}}$

20. $y = \dfrac{1}{x^2}$

21. $y = \dfrac{1}{\sqrt{1+x}}$

22. $f(t) = \dfrac{t^2-3}{t^2+3}$

23. How should the function $f(x) = x \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?

24. How should the function $g(x) = x^2 \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?

25. Where does $h(x) = |x^2 + 3x + 2|$ fail to be differentiable?

26. Using a calculator, find the slope of the secant line to $y = x^3 - 2x$ passing through the points corresponding to $x = 1$ and $x = 1 + \Delta x$, for several values of Δx of decreasing size, say $\Delta x = \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$. (Make a table.) Also, calculate $\dfrac{d}{dx}\left(x^3 - 2x\right)\Big|_{x=1}$ using the definition of derivative.

27. Repeat Exercise 26 for the function $f(x) = \dfrac{1}{x}$ and the points $x = 2$ and $x = 2 + \Delta x$.

Using the definition of derivative, find equations for the tangent lines to the curves in Exercises 28–31 at the points indicated.

28. $y = 5 + 4x - x^2$ at the point where $x = 2$

29. $y = \sqrt{x+6}$ at the point $(3, 3)$

30. $y = \dfrac{t}{t^2-2}$ at the point where $t = -2$

31. $y = \dfrac{2}{t^2+t}$ at the point where $t = a$

Calculate the derivatives of the functions in Exercises 32–37 using the General Power Rule. Where is each derivative valid?

32. $f(x) = x^{-17}$

33. $g(t) = t^{22}$

34. $y = x^{1/3}$

35. $y = x^{-1/3}$

36. $t^{-2.25}$

37. $s^{119/4}$

In Exercises 38–48, you may use the formulas for derivatives established in this section.

38. Calculate $\dfrac{d}{ds}\sqrt{s}\Big|_{s=9}$.

39. Find $F'(\frac{1}{4})$ if $F(x) = \dfrac{1}{x}$.

40. Find $f'(8)$ if $f(x) = x^{-2/3}$.

41. Find $dy/dt\Big|_{t=4}$ if $y = t^{1/4}$.

42. Find an equation of the straight line tangent to the curve $y = \sqrt{x}$ at $x = x_0$.

43. Find an equation of the straight line normal to the curve $y = 1/x$ at the point where $x = a$.

44. Show that the curve $y = x^2$ and the straight line $x + 4y = 18$ intersect at right angles at one of their two intersection points. *Hint:* find the product of their slopes at their intersection points.

45. There are two distinct straight lines that pass through the point $(1, -3)$ and are tangent to the curve $y = x^2$. Find their equations. *Hint:* draw a sketch. The points of tangency are not given; let them be denoted (a, a^2).

46. Find equations of two straight lines that have slope -2 and are tangent to the graph of $y = 1/x$.

47. Find the slope of a straight line that passes through the point $(-2, 0)$ and is tangent to the curve $y = \sqrt{x}$.

* **48.** Show that there are two distinct tangent lines to the curve $y = x^2$ passing through the point (a, b) provided $b < a^2$. How many tangent lines to $y = x^2$ pass through (a, b) if $b = a^2$? If $b > a^2$?

* **49.** Show that the derivative of an odd differentiable function is even and that the derivative of an even differentiable function is odd.

* **50.** Prove the case $r = -n$ (n is a positive integer) of the General Power Rule; that is, prove that

$$\frac{d}{dx} x^{-n} = -n\, x^{-n-1}.$$

Use the factorization of a difference of nth powers given in this section.

* **51.** Use the factoring of a difference of cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$

to help you calculate the derivative of $f(x) = x^{1/3}$ directly from the definition of derivative.

* **52.** Prove the General Power Rule for $\frac{d}{dx}x^r$, where $r = 1/n$, n being a positive integer. (*Hint:*

$$\begin{aligned}\frac{d}{dx}x^{1/n} &= \lim_{h\to 0}\frac{(x+h)^{1/n} - x^{1/n}}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^{1/n} - x^{1/n}}{((x+h)^{1/n})^n - (x^{1/n})^n}.\end{aligned}$$

Apply the factorization of the difference of nth powers to the denominator of the latter quotient.)

53. Give a proof of the power rule $\frac{d}{dx}x^n = nx^{n-1}$ for positive integers n using the Binomial Theorem:

$$\begin{aligned}(x+h)^n = x^n &+ \frac{n}{1}x^{n-1}h + \frac{n(n-1)}{1\times 2}x^{n-2}h^2\\ &+ \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-3}h^3 + \cdots + h^n.\end{aligned}$$

* **54.** Use right and left derivatives, $f'_+(a)$ and $f'_-(a)$, to define the concept of a half-line starting at $(a, f(a))$ being a right or left tangent to the graph of f at $x = a$. Show that the graph has a tangent line at $x = a$ if and only if it has right and left tangents that are opposite halves of the same straight line. What are the left and right tangents to the graphs of $y = x^{1/3}$, $y = x^{2/3}$, and $y = |x|$ at $x = 0$?

2.3 Differentiation Rules

If every derivative had to be calculated directly from the definition of derivative as in the examples of Section 2.2, calculus would indeed be a painful subject. Fortunately there is an easier way. We will develop several general *differentiation rules* that enable us to calculate the derivatives of complicated combinations of functions easily if we already know the derivatives of the elementary functions from which they are constructed. For instance, we will be able to find the derivative of $\dfrac{x^2}{\sqrt{x^2+1}}$ if we know the derivatives of x^2 and $\sqrt{x}$. The rules we develop in this section tell us how to differentiate sums, constant multiples, products, and quotients of functions whose derivatives we already know. In Section 2.4 we will learn how to differentiate composite functions.

Before developing these differentiation rules we need to establish one obvious but very important theorem which states, roughly, that the graph of a function cannot possibly have a break at a point where it is smooth.

THEOREM 1 **Differentiability implies continuity**

If f is differentiable at x, then f is continuous at x.

PROOF Since f is differentiable at x, we know that

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

exists. In order to prove that f is continuous at x, we need to show that

$$\lim_{h\to 0} f(x+h) = f(x).$$

Using the limit rules (Theorem 2 of Section 1.2), we have

$$\begin{aligned}\lim_{h\to 0} f(x+h) &= \lim_{h\to 0}\left(f(x) + \frac{f(x+h)-f(x)}{h} \times h\right)\\ &= f(x) + f'(x)\times 0 = f(x).\end{aligned}$$

Sums and Constant Multiples

The derivative of a sum (or difference) of functions is the sum (or difference) of the derivatives of those functions. The derivative of a constant multiple of a function is the same constant multiple of the derivative of the function.

THEOREM 2 **Differentiation rules for sums, differences, and constant multiples**

If functions f and g are differentiable at x, and if C is a constant, then the functions $f+g$, $f-g$, and Cf are all differentiable at x and

$$\begin{aligned}(f+g)'(x) &= f'(x) + g'(x),\\ (f-g)'(x) &= f'(x) - g'(x),\\ (Cf)'(x) &= Cf'(x).\end{aligned}$$

PROOF The proofs of all three assertions are straightforward, using the corresponding limit rules from Theorem 2 of Section 1.2. For the sum, we have

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}\\ &= \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}\\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\ &= f'(x)+g'(x),\end{aligned}$$

because the limit of a sum is the sum of the limits. The proof for the difference $f-g$ is similar. For the constant multiple, we have

$$\begin{aligned}(Cf)'(x) &= \lim_{h\to 0}\frac{Cf(x+h)-Cf(x)}{h}\\ &= C\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = Cf'(x).\end{aligned}$$

The rule for differentiating sums extends to sums of any finite number of terms:

$$(f_1+f_2+\cdots+f_n)' = f_1'+f_2'+\cdots+f_n'. \qquad (*)$$

To see this we can use a technique called **mathematical induction**. (See the note in the margin.) Theorem 2 shows that the case $n=2$ is true; this is STEP 1. For STEP 2, we must show that if the formula $(*)$ holds for some integer $n=k\ge 2$, then it must also hold for $n=k+1$. Therefore, assume that

$$(f_1+f_2+\cdots+f_k)' = f_1'+f_2'+\cdots+f_k'.$$

Then we have

$$\begin{aligned}&(f_1+f_2+\cdots+f_k+f_{k+1})'\\ &\quad= \big(\underbrace{(f_1+f_2+\cdots+f_k)}_{\text{Let this function be } f}+f_{k+1}\big)'\\ &\quad= (f+f_{k+1})' \qquad \text{(Now use the known case } n=2.)\\ &\quad= f'+f_{k+1}'\\ &\quad= f_1'+f_2'+\cdots+f_k'+f_{k+1}'.\end{aligned}$$

With both steps verified, we can claim that $(*)$ holds for any $n\ge 2$ *by induction*. In particular, therefore, the derivative of any polynomial is the sum of the derivatives of its terms.

Mathematical Induction

Mathematical induction is a technique for proving that a statement about an integer n is true for every integer n greater than or equal to some lowest integer n_0. The proof requires us to carry out two steps:

STEP 1. Prove that the statement is true for $n=n_0$.

STEP 2. Prove that if the statement is true for some integer $n=k$, where $k\ge n_0$, then it is also true for the next larger integer, $n=k+1$.

Step 2 prevents there from being a smallest integer greater than n_0 for which the statement is false. Being true for n_0, the statement must therefore be true for all larger integers.

Example 1 Calculate the derivatives of the functions:

(a) $2x^3 - 5x^2 + 4x + 7$, (b) $f(x) = 5\sqrt{x} + \dfrac{3}{x} - 18$, (c) $y = \dfrac{1}{7}t^4 - 3t^{7/3}$.

Solution Each of these functions is a sum of constant multiples of functions that we already know how to differentiate.

(a) $\dfrac{d}{dx}(2x^3 - 5x^2 + 4x + 7) = 2(3x^2) - 5(2x) + 4(1) + 0 = 6x^2 - 10x + 4.$

(b) $f'(x) = 5\left(\dfrac{1}{2\sqrt{x}}\right) + 3\left(-\dfrac{1}{x^2}\right) - 18(0) = \dfrac{5}{2\sqrt{x}} - \dfrac{3}{x^2}.$

(c) $\dfrac{dy}{dt} = \dfrac{1}{7}(4t^3) - 3\left(\dfrac{7}{3}t^{4/3}\right) = \dfrac{4}{7}t^3 - 7t^{4/3}.$

Example 2 Find an equation of the tangent to the curve $y = \dfrac{3x^3 - 4}{x}$ at the point on the curve where $x = -2$.

Solution If $x = -2$, then $y = 14$. The slope of the curve at $(-2, 14)$ is

$$\left.\frac{dy}{dx}\right|_{x=-2} = \left.\frac{d}{dx}\left(3x^2 - \frac{4}{x}\right)\right|_{x=-2} = \left.\left(6x + \frac{4}{x^2}\right)\right|_{x=-2} = -11.$$

An equation of the tangent line is $y = 14 - 11(x + 2)$, or $y = -11x - 8$.

The Product Rule

The rule for differentiating a product of functions is a little more complicated than that for sums. It is *not* true that the derivative of a product is the product of the derivatives.

THEOREM 3

The Product Rule

If functions f and g are differentiable at x, then their product fg is also differentiable at x, and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

PROOF We set up the Newton quotient for fg and then add 0 to the numerator in a way that enables us to involve the Newton quotients for f and g separately:

$$\begin{aligned}(fg)'(x) &= \lim_{h\to0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\\ &= \lim_{h\to0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}\\ &= \lim_{h\to0}\left(\frac{f(x+h) - f(x)}{h}g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}\right)\\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

To get the last line we have used the fact that f and g are differentiable and the fact that g is therefore continuous (Theorem 1), as well as limit rules from Theorem 2

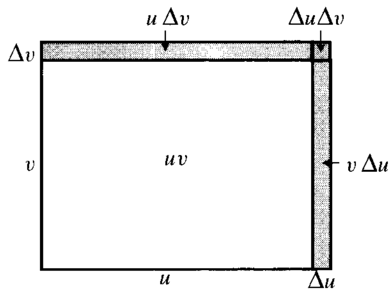

Figure 2.19

A graphical proof of the Product Rule

Here $u = f(x)$ and $v = g(x)$, so that the rectangular area uv represents $f(x)g(x)$. If x changes by an amount Δx, the corresponding increments in u and v are Δu and Δv. The change in the area of the rectangle is

$$\begin{aligned}&\Delta(uv)\\ &= (u + \Delta u)(v + \Delta v) - uv\\ &= (\Delta u)v + u(\Delta v) + (\Delta u)(\Delta v),\end{aligned}$$

the sum of the three shaded areas. Dividing by Δx and taking the limit as $\Delta x \to 0$, we get

$$\frac{d}{dx}(uv) = \left(\frac{du}{dx}\right)v + u\left(\frac{dv}{dx}\right),$$

since

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\Delta v = \frac{du}{dx} \times 0 = 0.$$

of Section 1.2. A graphical proof of the Product Rule is suggested by Figure 2.19.

Example 3 Find the derivative of $(x^2+1)(x^3+4)$ using and without using the Product Rule.

Solution Using the Product Rule with $f(x) = x^2 + 1$ and $g(x) = x^3 + 4$, we calculate

$$\frac{d}{dx}\big((x^2+1)(x^3+4)\big) = 2x(x^3+4) + (x^2+1)(3x^2) = 5x^4 + 3x^2 + 8x.$$

On the other hand, we can calculate the derivative by first multiplying the two binomials and then differentiating the resulting polynomial:

$$\frac{d}{dx}\big((x^2+1)(x^3+4)\big) = \frac{d}{dx}(x^5 + x^3 + 4x^2 + 4) = 5x^4 + 3x^2 + 8x.$$

■

Example 4 Find $\dfrac{dy}{dx}$ if $y = \left(2\sqrt{x} + \dfrac{3}{x}\right)\left(3\sqrt{x} - \dfrac{2}{x}\right)$.

Solution Applying the Product Rule with f and g being the two functions enclosed in the large parentheses, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{1}{\sqrt{x}} - \frac{3}{x^2}\right)\left(3\sqrt{x} - \frac{2}{x}\right) + \left(2\sqrt{x} + \frac{3}{x}\right)\left(\frac{3}{2\sqrt{x}} + \frac{2}{x^2}\right)\\ &= 6 - \frac{5}{2x^{3/2}} + \frac{12}{x^3}.\end{aligned}$$

■

Example 5 Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if $u(2) = 2$, $u'(2) = -5$, $v(2) = 1$, and $v'(2) = 3$.

Solution From the Product Rule we have

$$y' = (uv)' = u'v + uv'.$$

Therefore,

$$y'(2) = u'(2)v(2) + u(2)v'(2) = (-5)(1) + (2)(3) = -5 + 6 = 1.$$

■

Example 6 Use mathematical induction to verify the formula $\frac{d}{dx}x^n = n\,x^{n-1}$ for all positive integers n.

Solution For $n = 1$ the formula says that $\frac{d}{dx}x^1 = 1 = 1x^0$, so the formula is true in this case. We must show that if the formula is true for $n = k \geq 1$, then it is also true for $n = k + 1$. Therefore, assume that

$$\frac{d}{dx}x^k = kx^{k-1}.$$

Using the Product Rule we calculate

$$\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x^k x) = (kx^{k-1})(x) + (x^k)(1) = (k+1)x^k = (k+1)x^{(k+1)-1}.$$

Thus the formula is true for $n = k + 1$ also. The formula is true for all integers $n \geq 1$ *by induction.*

The Product Rule can be extended to products of any number of factors, for instance:

$$\begin{aligned}(fgh)'(x) &= f'(x)(gh)(x) + f(x)(gh)'(x)\\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).\end{aligned}$$

In general, the derivative of a product of n functions will have n terms; each term will be the same product but with one of the factors replaced by its derivative:

$$(f_1 f_2 f_3 \cdots f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 f_2 f_3 \cdots f_n'.$$

This can be proved by mathematical induction. See Exercise 54 at the end of this section.

The Reciprocal Rule

THEOREM 4

The Reciprocal Rule

If f is differentiable at x and $f(x) \neq 0$, then $1/f$ is differentiable at x, and

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}.$$

PROOF Using the definition of the derivative, we calculate

$$\begin{aligned}\frac{d}{dx}\frac{1}{f(x)} &= \lim_{h\to 0}\frac{\dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}}{h}\\ &= \lim_{h\to 0}\frac{f(x) - f(x+h)}{hf(x+h)f(x)}\\ &= \lim_{h\to 0}\left(\frac{-1}{f(x+h)f(x)}\right)\frac{f(x+h) - f(x)}{h}\\ &= \frac{-1}{(f(x))^2}f'(x)\end{aligned}$$

Again we have to use the continuity of f (Theorem 1) and the limit rules from Section 1.2.

Example 7 Differentiate the functions

(a) $\dfrac{1}{x^2+1}$ and (b) $f(t) = \dfrac{1}{t + \dfrac{1}{t}}$.

Solution Using the Reciprocal Rule:

(a) $\dfrac{d}{dx}\left(\dfrac{1}{x^2+1}\right) = \dfrac{-2x}{(x^2+1)^2}$.

(b) $f'(t) = \dfrac{-1}{\left(t+\dfrac{1}{t}\right)^2}\left(1 - \dfrac{1}{t^2}\right) = \dfrac{-t^2}{(t^2+1)^2}\,\dfrac{t^2-1}{t^2} = \dfrac{1-t^2}{(t^2+1)^2}$.

■

We can use the Reciprocal Rule to confirm the General Power Rule for negative integers:

$$\frac{d}{dx}x^{-n} = -n\,x^{-n-1},$$

since we have already proved the rule for positive integers. We have

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n} = \frac{-n\,x^{n-1}}{(x^n)^2} = -n\,x^{-n-1}.$$

Example 8 **(Differentiating sums of reciprocals)**

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2+x+1}{x^3}\right) &= \frac{d}{dx}\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right)\\ &= \frac{d}{dx}(x^{-1}+x^{-2}+x^{-3})\\ &= -x^{-2} - 2x^{-3} - 3x^{-4} = -\frac{1}{x^2} - \frac{2}{x^3} - \frac{3}{x^4}.\end{aligned}$$

■

The Quotient Rule

The Product Rule and the Reciprocal Rule can be combined to provide a rule for differentiating a quotient of two functions. Observe that

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\frac{1}{g(x)}\right) &= f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{(g(x))^2}\right)\\ &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.\end{aligned}$$

Thus we have proved the following Quotient Rule.

THEOREM 5

The Quotient Rule

If f and g are differentiable at x, and if $g(x) \neq 0$, then the quotient f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Sometimes students have trouble remembering this rule. (Getting the order of the terms in the numerator wrong will reverse the sign.) Try to remember (and use) the Quotient Rule in the following form:

$$(\text{quotient})' = \frac{(\text{denominator}) \times (\text{numerator})' - (\text{numerator}) \times (\text{denominator})'}{(\text{denominator})^2}$$

Example 9 Find the derivatives of

(a) $y = \dfrac{1 - x^2}{1 + x^2}$, (b) $\dfrac{\sqrt{t}}{3 - 5t}$, and (c) $f(\theta) = \dfrac{a + b\theta}{m + n\theta}$.

Solution We use the Quotient Rule in each case.

(a) $$\frac{dy}{dx} = \frac{(1 + x^2)(-2x) - (1 - x^2)(2x)}{(1 + x^2)^2} = -\frac{4x}{(1 + x^2)^2}.$$

(b) $$\frac{d}{dt}\left(\frac{\sqrt{t}}{3 - 5t}\right) = \frac{(3 - 5t)\dfrac{1}{2\sqrt{t}} - \sqrt{t}(-5)}{(3 - 5t)^2} = \frac{3 + 5t}{2\sqrt{t}(3 - 5t)^2}.$$

(c) $$f'(\theta) = \frac{(m + n\theta)(b) - (a + b\theta)(n)}{(m + n\theta)^2} = \frac{mb - na}{(m + n\theta)^2}.$$

In all three parts of Example 9 the Quotient Rule yielded fractions with numerators that were complicated but could be simplified algebraically. It is advisable to attempt such simplifications when calculating derivatives; the usefulness of derivatives in applications of calculus often depends on such simplifications.

Example 10 Find equations of any lines that pass through the point $(-1, 0)$ and are tangent to the curve $y = (x - 1)/(x + 1)$.

Solution The point $(-1, 0)$ does not lie on the curve, so it is not the point of tangency. Suppose a line is tangent to the curve at $x = a$, so the point of tangency is $(a, (a - 1)/(a + 1))$. Note that a cannot be -1. The slope of the line must be

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2}\right|_{x=a} = \frac{2}{(a + 1)^2}.$$

If the line also passes through $(-1, 0)$, its slope must also be given by

$$\frac{\frac{a-1}{a+1} - 0}{a - (-1)} = \frac{a-1}{(a+1)^2}.$$

Equating these two expressions for the slope, we get an equation to solve for a:

$$\frac{a-1}{(a+1)^2} = \frac{2}{(a+1)^2} \qquad \Longrightarrow \qquad a - 1 = 2.$$

Thus $a = 3$, and the slope of the line is $2/4^2 = 1/8$. There is only one line through $(-1, 0)$ tangent to the given curve, and its equation is

$$y = 0 + \frac{1}{8}(x+1) \qquad \text{or} \qquad x - 8y + 1 = 0.$$

■

Remark Derivatives of quotients of functions where the denominator is a monomial, such as in Example 8, are usually easier to do by breaking the quotient into several fractions (as was done in that example) rather than by using the Quotient Rule.

Exercises 2.3

In Exercises 1–32, calculate the derivatives of the given functions. Simplify your answers whenever possible.

1. $y = 3x^2 - 5x - 7$

2. $y = 4x^{1/2} - \frac{5}{x}$

3. $f(x) = Ax^2 + Bx + C$

4. $f(x) = \frac{6}{x^3} + \frac{2}{x^2} - 2$

5. $z = \frac{s^5 - s^3}{15}$

6. $y = x^{45} - x^{-45}$

7. $g(t) = t^{1/3} + 2t^{1/4} + 3t^{1/5}$

8. $y = 3\sqrt[3]{t^2} - \frac{2}{\sqrt{t^3}}$

9. $u = \frac{3}{5}x^{5/3} - \frac{5}{3}x^{-3/5}$

10. $F(x) = (3x - 2)(1 - 5x)$

11. $y = \sqrt{x}\left(5 - x - \frac{x^2}{3}\right)$

12. $g(t) = \frac{1}{2t - 3}$

13. $y = \frac{1}{x^2 + 5x}$

14. $y = \frac{4}{3 - x}$

15. $f(t) = \frac{\pi}{2 - \pi t}$

16. $g(y) = \frac{2}{1 - y^2}$

17. $f(x) = \frac{1 - 4x^2}{x^3}$

18. $g(u) = \frac{u\sqrt{u} - 3}{u^2}$

19. $y = \frac{2 + t + t^2}{\sqrt{t}}$

20. $z = \frac{x - 1}{x^{2/3}}$

21. $f(x) = \frac{3 - 4x}{3 + 4x}$

22. $z = \frac{t^2 + 2t}{t^2 - 1}$

23. $s = \frac{1 + \sqrt{t}}{1 - \sqrt{t}}$

24. $f(x) = \frac{x^3 - 4}{x + 1}$

25. $f(x) = \frac{ax + b}{cx + d}$

26. $F(t) = \frac{t^2 + 7t - 8}{t^2 - t + 1}$

27. $f(x) = (1 + x)(1 + 2x)(1 + 3x)(1 + 4x)$

28. $f(r) = (r^{-2} + r^{-3} - 4)(r^2 + r^3 + 1)$

29. $y = (x^2 + 4)(\sqrt{x} + 1)(5x^{2/3} - 2)$

30. $y = \frac{(x^2 + 1)(x^3 + 2)}{(x^2 + 2)(x^3 + 1)}$

***31.** $y = \frac{x}{2x + \frac{1}{3x + 1}}$

***32.** $f(x) = \frac{(\sqrt{x} - 1)(2 - x)(1 - x^2)}{\sqrt{x}(3 + 2x)}$

Calculate the derivatives in Exercises 33–36, given that $f(2) = 2$ and $f'(2) = 3$.

33. $\left.\frac{d}{dx}\left(\frac{x^2}{f(x)}\right)\right|_{x=2}$

34. $\left.\frac{d}{dx}\left(\frac{f(x)}{x^2}\right)\right|_{x=2}$

35. $\left.\frac{d}{dx}\left(x^2 f(x)\right)\right|_{x=2}$

36. $\left.\frac{d}{dx}\left(\frac{f(x)}{x^2 + f(x)}\right)\right|_{x=2}$

37. Find $\displaystyle \frac{d}{dx}\left(\frac{x^2-4}{x^2+4}\right)\bigg|_{x=-2}$.

38. Find $\displaystyle \frac{d}{dt}\left(\frac{t(1+\sqrt{t})}{5-t}\right)\bigg|_{t=4}$.

39. If $f(x) = \dfrac{\sqrt{x}}{x+1}$, find $f'(2)$.

40. Find $\displaystyle \frac{d}{dt}\Big((1+t)(1+2t)(1+3t)(1+4t)\Big)\bigg|_{t=0}$.

41. Find an equation of the tangent line to $y = \dfrac{2}{3-4\sqrt{x}}$ at the point $(1, -2)$.

42. Find equations of the tangent and normal to $y = \dfrac{x+1}{x-1}$ at $x = 2$.

43. Find the points on the curve $y = x + 1/x$ where the tangent line is horizontal.

44. Find the equations of all horizontal lines that are tangent to the curve $y = x^2(4 - x^2)$.

45. Find the coordinates of all points where the curve $y = \dfrac{1}{x^2+x+1}$ has a horizontal tangent line.

46. Find the coordinates of points on the curve $y = \dfrac{x+1}{x+2}$ where the tangent line is parallel to the line $y = 4x$.

47. Find the equation of the straight line that passes through the point $(0, b)$ and is tangent to the curve $y = 1/x$. Assume $b \neq 0$.

* 48. Show that the curve $y = x^2$ intersects the curve $y = 1/\sqrt{x}$ at right angles.

49. Find two straight lines that are tangent to $y = x^3$ and pass through the point $(2, 8)$.

50. Find two straight lines that are tangent to $y = x^2/(x-1)$ and pass through the point $(2, 0)$.

51. **(A Square Root Rule)** Show that if f is differentiable at x and $f(x) > 0$, then

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}.$$

Use this Square Root Rule to find the derivative of $\sqrt{x^2+1}$.

52. Show that $f(x) = |x^3|$ is differentiable at every real number x, and find its derivative.

Mathematical Induction

53. Use mathematical induction to prove that $\dfrac{d}{dx}x^{n/2} = \dfrac{n}{2}x^{(n/2)-1}$ for every positive integer n. Then use the Reciprocal Rule to get the same result for negative integers n.

54. Use mathematical induction to prove the formula for the derivative of a product of n functions given earlier in this section.

2.4 The Chain Rule

Although we can differentiate $\sqrt{x}$ and $x^2 + 1$, we cannot yet differentiate $\sqrt{x^2+1}$. To do this, we need a rule that tells us how to differentiate *composites* of functions whose derivatives we already know. This rule is known as the Chain Rule and is the most often used of all the differentiation rules.

Example 1 The function $\dfrac{1}{x^2-4}$ is the composite $f(g(x))$ of $f(u) = \dfrac{1}{u}$ and $g(x) = x^2 - 4$, which have derivatives

$$f'(u) = \frac{-1}{u^2} \qquad \text{and} \qquad g'(x) = 2x.$$

According to the Reciprocal Rule (which is a special case of the Chain Rule),

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \frac{d}{dx}\left(\frac{1}{x^2-4}\right) = \frac{-2x}{(x^2-4)^2} = \frac{-1}{(x^2-4)^2}(2x)\\ &= f'(g(x))g'(x).\end{aligned}$$

■

This example suggests that the derivative of a composite function $f(g(x))$ is the derivative of f evaluated at $g(x)$ multiplied by the derivative of g evaluated at x.

This is the Chain Rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).$$

THEOREM 6

The Chain Rule

If $f(u)$ is differentiable at $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $f \circ g(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

In terms of Leibniz notation, if $y = f(u)$ where $u = g(x)$, then $y = f(g(x))$ and:

at u, y is changing $\frac{dy}{du}$ times as fast as u is changing;

at x, u is changing $\frac{du}{dx}$ times as fast as x is changing.

Therefore, at x, $y = f(u) = f(g(x))$ is changing $\frac{dy}{du} \times \frac{du}{dx}$ times as fast as x is changing. That is:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \qquad \text{where } \frac{dy}{du} \text{ is evaluated at } u = g(x).$$

It appears as though the symbol du cancels from the numerator and denominator, but this is not meaningful because dy/du was not defined as the quotient of two quantities, but rather as a single quantity, the derivative of y with respect to u.

We would like to prove Theorem 6 by writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$$

and taking the limit as $\Delta x \to 0$. Such a proof is valid for most composite functions but not all. (See Exercise 46 at the end of this section.) A correct proof will be given later in this section, but first we do more examples to give a better idea of how the Chain Rule works.

Example 2 Find the derivative of $y = \sqrt{x^2 + 1}$.

Solution Here $y = f(g(x))$, where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since the derivatives of f and g are

$$f'(u) = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x,$$

the Chain Rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)\\ &= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x) = \frac{x}{\sqrt{x^2+1}}.\end{aligned}$$

■

Outside and Inside Functions

In the composite $f(g(x))$, the function f is "outside," and the function g is "inside." The Chain Rule says that the derivative of the composite is the derivative f' of the outside function evaluated at the inside function $g(x)$, multiplied by the derivative $g'(x)$ of the inside function:
$\frac{d}{dx}f(g(x)) = f'(g(x)) \times g'(x)$.

Usually, when applying the Chain Rule, we do not introduce symbols to represent the functions being composed, but rather just proceed to calculate the derivative of the "outside" function and then multiply by the derivative of whatever is "inside." You can say to yourself: "the derivative of f of something is f' of that thing, multiplied by the derivative of that thing."

Example 3 Find derivatives of the following functions:

(a) $(7x-3)^{10}$, (b) $f(t) = |t^2-1|$, and (c) $\left(3x + \frac{1}{(2x+1)^3}\right)^{1/4}$.

Solution

(a) Here, the outside function is the 10th power; it must be differentiated first and the result multiplied by the derivative of the expression $7x-3$:

$$\frac{d}{dx}(7x-3)^{10} = 10(7x-3)^9(7) = 70(7x-3)^9.$$

(b) Here, we are differentiating the absolute value of something. The derivative is signum of that thing, multiplied by the derivative of that thing:

$$f'(t) = \big(\operatorname{sgn}(t^2-1)\big)(2t) = \frac{2t(t^2-1)}{|t^2-1|} = \begin{cases} 2t & \text{if } t < -1 \text{ or } t > 1 \\ -2t & \text{if } -1 < t < 1 \\ \text{undefined} & \text{if } t = \pm 1. \end{cases}$$

(c) Here, we will need to use the Chain Rule twice. We begin by differentiating the $1/4$ power of something, but the something involves the -3rd power of $2x+1$, and the derivative of that will also require the Chain Rule:

$$\begin{aligned} \frac{d}{dx}\left(3x + \frac{1}{(2x+1)^3}\right)^{1/4} &= \frac{1}{4}\left(3x + \frac{1}{(2x+1)^3}\right)^{-3/4} \frac{d}{dx}\left(3x + \frac{1}{(2x+1)^3}\right) \\ &= \frac{1}{4}\left(3x + \frac{1}{(2x+1)^3}\right)^{-3/4}\left(3 - \frac{3}{(2x+1)^4}\frac{d}{dx}(2x+1)\right) \\ &= \frac{3}{4}\left(1 - \frac{2}{(2x+1)^4}\right)\left(3x + \frac{1}{(2x+1)^3}\right)^{-3/4}. \end{aligned}$$

When you start to feel comfortable with the Chain Rule, you may want to "save a line or two" by carrying out the whole differentiation in one step:

$$\begin{aligned} \frac{d}{dx}\left(3x + \frac{1}{(2x+1)^3}\right)^{1/4} &= \frac{1}{4}\left(3x + \frac{1}{(2x+1)^3}\right)^{-3/4}\left(3 - \frac{3}{(2x+1)^4}(2)\right) \\ &= \frac{3}{4}\left(1 - \frac{2}{(2x+1)^4}\right)\left(3x + \frac{1}{(2x+1)^3}\right)^{-3/4}. \end{aligned}$$

■

Use of the Chain Rule produces products of factors that do not usually come out in the order you would naturally write them. Often you will want to rewrite the result with the factors in a different order. This is obvious in parts (a) and (c) of the example above. In monomials (expressions that are products of factors), it is common to write the factors in order of increasing complexity from left to right, with numerical factors coming first. One time when you would *not* waste time doing this, or trying to make any other simplification, is when you are going to evaluate the derivative at a particular number. In this case, substitute the number as soon as you have calculated the derivative, before doing any simplification:

$$\frac{d}{dx}(x^2-3)^{10}\bigg|_{x=2} = 10(x^2-3)^9(2x)\bigg|_{x=2} = (10)(1^9)(4) = 40.$$

Example 4 Suppose that f is a differentiable function on the real line. In terms of the derivative f' of f, express the derivatives of:

(a) $f(3x)$, (b) $f(x^2)$, (c) $f(\pi f(x))$, and (d) $[f(3-2f(x))]^4$.

Solution

(a) $\dfrac{d}{dx} f(3x) = \big(f'(3x)\big)(3) = 3f'(3x)$.

(b) $\dfrac{d}{dx} f(x^2) = \big(f'(x^2)\big)(2x) = 2xf'(x^2)$.

(c) $\dfrac{d}{dx} f(\pi f(x)) = \big(f'(\pi f(x))\big)(\pi f'(x)) = \pi f'(x)f'(\pi f(x))$.

(d) $$\begin{aligned}\frac{d}{dx}[f(3-2f(x))]^4 &= 4[f(3-2f(x))]^3 f'(3-2f(x))(-2f'(x)) \\ &= -8f'(x)f'(3-2f(x))[f(3-2f(x))]^3.\end{aligned}$$

■

Finding Derivatives with Maple

Computer algebra systems know the derivatives of elementary functions and can calculate the derivatives of combinations of these functions symbolically, using differentiation rules. Maple's `D` operator can be used to find the derivative function `D(f)` of a function `f` of one variable. Alternatively, you can use `diff` to differentiate an expression with respect to a variable and then use the substitution routine `subs` to evaluate the result at a particular number.

```
> f := x -> sqrt(1+2*x^2);
```

$$f := x \to \sqrt{1+2x^2}$$

```
> fprime := D(f);
```

$$fprime := x \to 2\frac{x}{\sqrt{1+2x^2}}$$

```
> fprime(2);
```

$$\frac{4}{3}$$

```
> diff(t^2*sin(3*t),t);
```

$$2\,t\,\sin(3\,t) + 3\,t^2\cos(3\,t)$$

```
> simplify(subs(t=Pi/12, %));
```

$$\frac{1}{12}\pi\sqrt{2} + \frac{1}{96}\pi^2\sqrt{2}$$

Building the Chain Rule into Differentiation Formulas

If u is a differentiable function of x and $y = u^n$, then the Chain Rule gives

$$\frac{d}{dx}u^n = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

The formula

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

is just the formula $\frac{d}{dx}x^n = nx^{n-1}$ with an application of the Chain Rule built in, so that it applies to functions of x rather than just to x. Some other differentiation rules with built-in Chain Rule applications are:

$$\frac{d}{dx}\left(\frac{1}{u}\right) = \frac{-1}{u^2}\frac{du}{dx} \qquad \text{(the Reciprocal Rule)}$$

$$\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}\frac{du}{dx} \qquad \text{(the Square Root Rule)}$$

$$\frac{d}{dx}u^r = r\,u^{r-1}\frac{du}{dx} \qquad \text{(the General Power Rule)}$$

$$\frac{d}{dx}|u| = \operatorname{sgn} u\,\frac{du}{dx} = \frac{u}{|u|}\frac{du}{dx} \qquad \text{(the Absolute Value Rule)}$$

Proof of the Chain Rule (Theorem 6)

Suppose that f is differentiable at the point $u = g(x)$ and that g is differentiable at x. Let the function $E(k)$ be defined by

$$E(0) = 0,$$
$$E(k) = \frac{f(u+k) - f(u)}{k} - f'(u), \qquad \text{if } k \neq 0.$$

By the definition of derivative, $\lim_{k\to 0} E(k) = f'(u) - f'(u) = 0 = E(0)$, so $E(k)$ is continuous at $k = 0$. Also, whether $k = 0$ or not, we have

$$f(u+k) - f(u) = \big(f'(u) + E(k)\big)k.$$

Now put $u = g(x)$ and $k = g(x+h) - g(x)$, so that $u + k = g(x+h)$, and obtain

$$f(g(x+h)) - f(g(x)) = \big(f'(g(x)) + E(k)\big)(g(x+h) - g(x)).$$

Since g is differentiable at x, $\lim_{h\to 0}[g(x+h) - g(x)]/h = g'(x)$. Also, g is continuous at x by Theorem 1, so $\lim_{h\to 0} k = \lim_{h\to 0}(g(x+h) - g(x)) = 0$. Since E is continuous at 0, $\lim_{h\to 0} E(k) = \lim_{k\to 0} E(k) = E(0) = 0$. Hence

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0}\big(f'(g(x)) + E(k)\big)\frac{g(x+h) - g(x)}{h} \\ &= \big(f'(g(x)) + 0\big)g'(x) = f'(g(x))g'(x),\end{aligned}$$

which was to be proved.

Exercises 2.4

Find the derivatives of the functions in Exercises 1–16.

1. $y = (2x+3)^6$

2. $y = \left(1 - \frac{x}{3}\right)^{99}$

3. $f(x) = (4 - x^2)^{10}$

4. $y = \sqrt{1 - 3x^2}$

5. $F(t) = \left(2 + \frac{3}{t}\right)^{-10}$

6. $(1 + x^{2/3})^{3/2}$

7. $\frac{3}{5 - 4x}$

8. $(1 - 2t^2)^{-3/2}$

9. $y = |1 - x^2|$

10. $f(t) = |2 + t^3|$

11. $y = 4x + |4x - 1|$

12. $y = (2 + |x|^3)^{1/3}$

13. $y = \frac{1}{2 + \sqrt{3x+4}}$

14. $f(x) = \left(1 + \sqrt{\frac{x-2}{3}}\right)^4$

15. $z = \left(u + \frac{1}{u-1}\right)^{-5/3}$

16. $y = \frac{x^5\sqrt{3 + x^6}}{(4 + x^2)^3}$

17. Sketch the graph of the function in Exercise 10.

18. Sketch the graph of the function in Exercise 11.

Verify that the General Power Rule holds for the functions in Exercises 19–21.

19. $x^{1/4} = \sqrt{\sqrt{x}}$

20. $x^{3/4} = \sqrt{x\sqrt{x}}$

21. $x^{3/2} = \sqrt{(x^3)}$

In Exercises 22–29, express the derivative of the given function in terms of the derivative f' of the differentiable function f.

22. $f(2t+3)$

23. $f(5x - x^2)$

24. $\left[f\left(\frac{2}{x}\right)\right]^3$

25. $\sqrt{3 + 2f(x)}$

26. $f\left(\sqrt{3 + 2t}\right)$

27. $f\left(3 + 2\sqrt{x}\right)$

28. $f\big(2f(3f(x))\big)$

29. $f\big(2 - 3f(4 - 5t)\big)$

30. Find $\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\Bigg|_{x=-2}$.

31. Find $\frac{d}{dt}\sqrt{3t-7}\Big|_{t=3}$.

32. If $f(x) = \frac{1}{\sqrt{2x+1}}$, find $f'(4)$.

33. If $y = (x^3 + 9)^{17/2}$, find $y'\Big|_{x=-2}$.

34. Find $F'(0)$ if $F(x) = (1+x)(2+x)^2(3+x)^3(4+x)^4$.

***35.** Calculate y' if $y = (x + ((3x)^5 - 2)^{-1/2})^{-6}$. Try to do it all in one step.

In Exercises 36–39, find an equation of the tangent line to the given curve at the given point.

36. $y = \sqrt{1 + 2x^2}$ at $x = 2$

37. $y = (1 + x^{2/3})^{3/2}$ at $x = -1$

38. $y = (ax + b)^8$ at $x = b/a$

39. $y = 1/(x^2 - x + 3)^{3/2}$ at $x = -2$

40. Show that the derivative of $f(x) = (x-a)^m(x-b)^n$ vanishes at some point between a and b if m and n are positive integers.

Use Maple or another computer algebra system to evaluate and simplify the derivatives of the functions in Exercises 41–44.

41. $y = \sqrt{x^2+1} + \frac{1}{(x^2+1)^{3/2}}$

42. $y = \frac{(x^2-1)(x^2-4)(x^2-9)}{x^6}$

43. $\frac{dy}{dt}\Big|_{t=2}$ if $y = (t+1)(t^2+2)(t^3+3)(t^4+4)(t^5+5)$

44. $f'(1)$ if $f(x) = \frac{(x^2+3)^{1/2}(x^3+7)^{1/3}}{(x^4+15)^{1/4}}$

45. Does the Chain Rule enable you to calculate the derivatives of $|x|^2$ and $|x^2|$ at $x = 0$? Do these functions have derivatives at $x = 0$? Why?

***46.** What is wrong with the following "proof" of the Chain Rule? Let $k = g(x+h) - g(x)$. Then $\lim_{h\to 0} k = 0$. Thus

$$\lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}$$
$$= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\,\frac{g(x+h) - g(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(g(x)+k) - f(g(x))}{k}\,\frac{g(x+h) - g(x)}{h}$$
$$= f'(g(x))\,g'(x).$$

2.5 Derivatives of Trigonometric Functions

The trigonometric functions, especially sine and cosine, play a very important role in the mathematical modelling of real-world phenomena. In particular, they arise whenever quantities fluctuate in a periodic way. Elastic motions, vibrations, and waves of all kinds naturally involve the trigonometric functions, and many physical and mechanical laws are formulated as differential equations having these functions as solutions.

In this section we will calculate the derivatives of the six trigonometric functions. We only have to work hard for one of them, sine; the others then follow from known identities and the differentiation rules of Section 2.3.

Some Special Limits

First, we have to establish some trigonometric limits that we will need to calculate the derivative of sine. It is assumed throughout that the arguments of the trigonometric functions are measured in radians.

THEOREM 7

The functions $\sin\theta$ and $\cos\theta$ are continuous at every value of θ. In particular, at $\theta = 0$ we have:

$$\lim_{\theta\to 0} \sin\theta = \sin 0 = 0 \quad \text{and} \quad \lim_{\theta\to 0} \cos\theta = \cos 0 = 1.$$

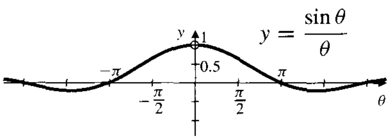

Figure 2.20 It appears that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$

This result is obvious from the graphs of sine and cosine, so we will not prove it here. A proof can be based on the Squeeze Theorem (Theorem 4 of Section 1.2). The method is suggested in Exercise 62 at the end of this section.

The graph of the function $y = (\sin\theta)/\theta$ is shown in Figure 2.20. Although it is not defined at $\theta = 0$, this function appears to have limit 1 as θ approaches 0.

THEOREM 8

An important trigonometric limit

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad \text{(where } \theta \text{ is in radians).}$$

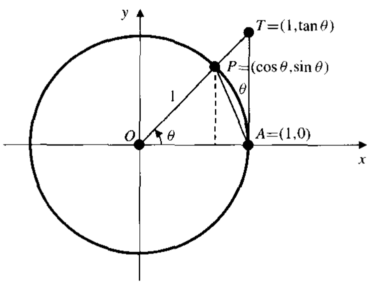

Figure 2.21 Area $\triangle\, OAP$ < Area sector OAP < Area $\triangle\, OAT$

PROOF Let $0 < \theta < \pi/2$, and represent θ as shown in Figure 2.21. Points $A(1, 0)$ and $P(\cos\theta, \sin\theta)$ lie on the unit circle $x^2 + y^2 = 1$. The area of the circular sector OAP lies between the areas of triangles OAP and OAT:

$$\text{Area } \triangle\, OAP < \text{Area sector } OAP < \text{Area } \triangle\, OAT.$$

As shown in Section P.6, the area of a circular sector having central angle θ (radians) and radius 1 is $\theta/2$. The area of a triangle is $(1/2) \times \text{base} \times \text{height}$, so

$$\text{Area } \triangle\, OAP = \frac{1}{2}(1)(\sin\theta) = \frac{\sin\theta}{2},$$
$$\text{Area } \triangle\, OAT = \frac{1}{2}(1)(\tan\theta) = \frac{\sin\theta}{2\cos\theta}.$$

Thus

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\sin\theta}{2\cos\theta},$$

or, upon multiplication by the positive number $2/\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Now take reciprocals, thereby reversing the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0+}\cos\theta = 1$ by Theorem 7, the Squeeze Theorem gives

$$\lim_{\theta\to 0+}\frac{\sin\theta}{\theta} = 1.$$

Finally, note that $\sin\theta$ and θ are *odd functions*. Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*: $f(-\theta) = f(\theta)$, as shown in Figure 2.20. This symmetry implies that the left limit at 0 must have the same value as the right limit:

$$\lim_{\theta\to 0-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem 1 of Section 1.2.

Theorem 8 can be combined with limit rules and known trigonometric identities to yield other trigonometric limits.

Example 1 Show that $\lim_{h\to 0}\dfrac{\cos h - 1}{h} = 0.$

Solution Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} \qquad \text{Let } \theta = h/2.\\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta = -(1)(0) = 0.\end{aligned}$$

The Derivatives of Sine and Cosine

To calculate the derivative of $\sin x$ we need the addition formula for sine (see Section P.6):

$$\sin(x+h) = \sin x\,\cos h + \cos x\,\sin h.$$

THEOREM 9

The derivative of the sine function is the cosine function.

$$\frac{d}{dx}\sin x = \cos x$$

PROOF We use the definition of derivative, the addition formula for sine, the rules for combining limits, Theorem 8, and the result of Example 1:

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\ &= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}\\ &= (\sin x)\cdot(0)+(\cos x)\cdot(1)=\cos x.\end{aligned}$$

THEOREM 10 **The derivative of the cosine function is minus the sine function.**

$$\frac{d}{dx}\cos x = -\sin x$$

PROOF We could mimic the proof for sine above, using the addition rule for cosine, $\cos(x+h)=\cos x\cos h-\sin x\sin h$. An easier way is to make use of the complementary angle identities, $\sin((\pi/2)-x)=\cos x$ and $\cos((\pi/2)-x)=\sin x$, and the Chain Rule from Section 2.4:

$$\frac{d}{dx}\cos x=\frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right)=(-1)\cos\left(\frac{\pi}{2}-x\right)=-\sin x.$$

Notice the minus sign in the derivative of cosine. The derivative of the sine is the cosine, but the derivative of the cosine is *minus* the sine. This is shown graphically in Figure 2.22.

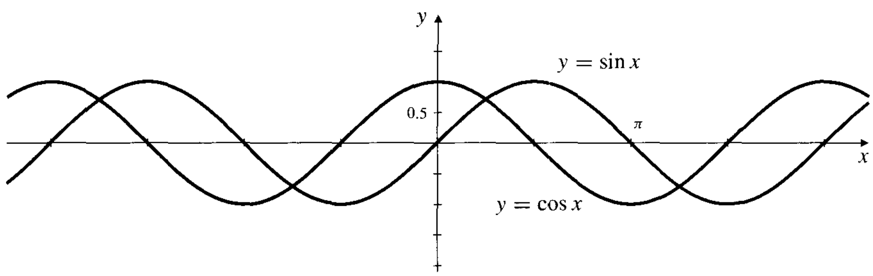

Figure 2.22 The sine and cosine plotted together. The slope of the sine curve at x is $\cos x$; the slope of the cosine curve at x is $-\sin x$.

Example 2 Evaluate the derivatives of the following functions:

(a) $\sin(\pi x)+\cos(3x)$, (b) $x^2\sin\sqrt{x}$, and (c) $\dfrac{\cos x}{1-\sin x}$.

Solution

(a) By the Sum Rule and the Chain Rule:

$$\frac{d}{dx}(\sin(\pi x)+\cos(3x))=\cos(\pi x)(\pi)-\sin(3x)(3)=\pi\cos(\pi x)-3\sin(3x).$$

(b) By the Product and Chain Rules:

$$\frac{d}{dx}(x^2 \sin \sqrt{x}) = 2x \sin \sqrt{x} + x^2 \left(\cos \sqrt{x}\right) \frac{1}{2\sqrt{x}} = 2x \sin \sqrt{x} + \frac{1}{2} x^{3/2} \cos \sqrt{x}.$$

(c) By the Quotient Rule:

$$\begin{aligned}\frac{d}{dx}\left(\frac{\cos x}{1-\sin x}\right) &= \frac{(1-\sin x)(-\sin x) - (\cos x)(0-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\ &= \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}\end{aligned}$$

We used the identity $\sin^2 x + \cos^2 x = 1$ to simplify the middle line. ∎

Using trigonometric identities can sometimes change the way a derivative is calculated. Carrying out a differentiation in different ways can lead to different-looking answers, but they should be equal if no errors have been made.

Example 3 Use two different methods to find the derivative of the function $f(t) = \sin t \cos t$.

Solution By the Product Rule:

$$f'(t) = (\cos t)(\cos t) + (\sin t)(-\sin t) = \cos^2 t - \sin^2 t.$$

On the other hand, since $\sin(2t) = 2 \sin t \cos t$, we have

$$f'(t) = \frac{d}{dt}\left(\frac{1}{2}\sin(2t)\right) = \left(\frac{1}{2}\right)(2)\cos(2t) = \cos(2t).$$

The two answers are really the same, since $\cos(2t) = \cos^2 t - \sin^2 t$. ∎

It is very important to remember that the formulas for the derivatives of $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in *radians*. Since $180° = \pi$ radians, $x° = \pi x/180$ radians. By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

Similarly, the derivative of $\cos(x°)$ is $-(\pi/180)\sin(x°)$.

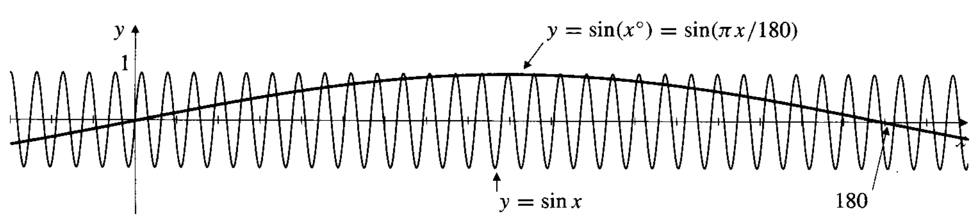

Figure 2.23 $\sin(x°)$ oscillates much more slowly than $\sin x$. Its maximum slope is $\pi/180$

Continuity

The six trigonometric functions are differentiable, and therefore continuous (by Theorem 1), everywhere on their domains. This means that we can calculate the limits of most trigonometric functions as $x \to a$ by evaluating them at $x = a$.

The Derivatives of the Other Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable everywhere, the functions

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined (i.e., where their denominators are not zero). Their derivatives can be calculated by the Quotient and Reciprocal Rules and are as follows:

The three "co-" functions (cosine, cotangent, and cosecant) have explicit minus signs in their derivatives.

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x.$$

Example 4 Verify the derivative formulas for $\tan x$ and $\sec x$.

Solution We use the Quotient Rule for tangent and the Reciprocal Rule for secant:

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x. \\
\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{-1}{\cos^2 x}\frac{d}{dx}(\cos x) \\
&= \frac{-1}{\cos^2 x}(-\sin x) = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \\
&= \sec x\ \tan x.
\end{aligned}$$

■

Example 5

(a) $\frac{d}{dx}\left[3x + \cot\left(\frac{x}{2}\right)\right] = 3 + \left[-\csc^2\left(\frac{x}{2}\right)\right]\frac{1}{2} = 3 - \frac{1}{2}\csc^2\left(\frac{x}{2}\right)$

(b) $$\begin{aligned}\frac{d}{dx}\left(\frac{3}{\sin(2x)}\right) &= \frac{d}{dx}(3\csc(2x)) \\ &= 3(-\csc(2x)\cot(2x))(2) = -6\csc(2x)\cot(2x).\end{aligned}$$

■

Example 6 Find the tangent and normal lines to the curve $y = \tan(\pi x/4)$ at the point $(1, 1)$.

Solution The slope of the tangent to $y = \tan(\pi x/4)$ at $(1, 1)$ is:

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{\pi}{4}\sec^2(\pi x/4)\bigg|_{x=1} = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\left(\sqrt{2}\right)^2 = \frac{\pi}{2}.$$

The tangent is the line

$$y = 1 + \frac{\pi}{2}(x-1), \qquad \text{or} \qquad y = \frac{\pi x}{2} - \frac{\pi}{2} + 1.$$

The normal has slope $m = -2/\pi$, so its point-slope equation is

$$y = 1 - \frac{2}{\pi}(x-1), \qquad \text{or} \qquad y = -\frac{2x}{\pi} + \frac{2}{\pi} + 1.$$

■

Exercises 2.5

1. Verify the formula for the derivative of $\csc x = 1/(\sin x)$.
2. Verify the formula for the derivative of $\cot x = (\cos x)/(\sin x)$.

Find the derivatives of the functions in Exercises 3–36. Simplify your answers whenever possible. Also be on the lookout for ways you might simplify the given expression before differentiating it.

3. $y = \cos 3x$
4. $y = \sin \dfrac{x}{5}$
5. $y = \tan \pi x$
6. $y = \sec ax$
7. $y = \cot(4 - 3x)$
8. $y = \sin((\pi - x)/3)$
9. $f(x) = \cos(s - rx)$
10. $y = \sin(Ax + B)$
11. $\sin(\pi x^2)$
12. $\cos(\sqrt{x})$
13. $y = \sqrt{1 + \cos x}$
14. $\sin(2\cos x)$
15. $f(x) = \cos(x + \sin x)$
16. $g(\theta) = \tan(\theta \sin \theta)$
17. $u = \sin^3(\pi x/2)$
18. $y = \sec(1/x)$
19. $F(t) = \sin at \cos at$
20. $G(\theta) = \dfrac{\sin a\theta}{\cos b\theta}$
21. $\sin(2x) - \cos(2x)$
22. $\cos^2 x - \sin^2 x$
23. $\tan x + \cot x$
24. $\sec x - \csc x$
25. $\tan x - x$
26. $\tan(3x)\cot(3x)$
27. $t \cos t - \sin t$
28. $t \sin t + \cos t$
29. $\dfrac{\sin x}{1 + \cos x}$
30. $\dfrac{\cos x}{1 + \sin x}$
31. $x^2 \cos(3x)$
32. $g(t) = \sqrt{(\sin t)/t}$
33. $v = \sec(x^2)\tan(x^2)$
34. $z = \dfrac{\sin\sqrt{x}}{1 + \cos\sqrt{x}}$
35. $\sin(\cos(\tan t))$
36. $f(s) = \cos(s + \cos(s + \cos s))$
37. Given that $\sin 2x = 2\sin x \cos x$, deduce that $\cos 2x = \cos^2 x - \sin^2 x$.
38. Given that $\cos 2x = \cos^2 x - \sin^2 x$, deduce that $\sin 2x = 2\sin x \cos x$.

In Exercises 39–42, find equations for the lines that are tangent and normal to the curve $y = f(x)$ at the given point.

39. $y = \sin x$, $(\pi, 0)$
40. $y = \tan(2x)$, $(0, 0)$
41. $y = \sqrt{2}\cos(x/4)$, $(\pi, 1)$
42. $y = \cos^2 x$, $\left(\dfrac{\pi}{3}, \dfrac{1}{4}\right)$
43. Find an equation of the line tangent to the curve $y = \sin(x°)$ at the point where $x = 45$.
44. Find an equation of the straight line normal to $y = \sec(x°)$ at the point where $x = 60$.
45. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$.
46. Find the points on the curve $y = \tan(2x)$, $-\pi/4 < x < \pi/4$, where the normal is parallel to the line $y = -x/8$.
47. Show that the graphs of $y = \sin x$, $y = \cos x$, $y = \sec x$, and $y = \csc x$ have horizontal tangents.
48. Show that the graphs of $y = \tan x$ and $y = \cot x$ never have horizontal tangents.

Do the graphs of the functions in Exercises 49–52 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not?

49. $y = x + \sin x$ **50.** $y = 2x + \sin x$

51. $y = x + 2\sin x$ **52.** $y = x + 2\cos x$

Find the limits in Exercises 53–56.

53. $\lim_{x\to 0} \frac{\tan(2x)}{x}$ **54.** $\lim_{x\to\pi} \sec(1 + \cos x)$

55. $\lim_{x\to 0} (x^2 \csc x \cot x)$ **56.** $\lim_{x\to 0} \cos\left(\frac{\pi - \pi\cos^2 x}{x^2}\right)$

57. Use the method of Example 1 to evaluate $\lim_{h\to 0} \frac{1 - \cos h}{h^2}$.

58. Find values of a and b that make

$$f(x) = \begin{cases} ax + b, & x < 0 \\ 2\sin x + 3\cos x, & x \ge 0 \end{cases}$$

differentiable at $x = 0$.

59. How many straight lines that pass through the origin are tangent to $y = \cos x$? Find (to 6 decimal places) the slopes of the two such lines that have the largest positive slopes.

Use Maple or another computer algebra system to evaluate and simplify the derivatives of the functions in Exercises 60–61.

60. $\left.\frac{d}{dx} \frac{x\cos(x\sin x)}{x + \cos(x\cos x)}\right|_{x=0}$

61. $\left.\frac{d}{dx}\left(\sqrt{2x^2+3}\sin(x^2) - \frac{(2x^2+3)^{3/2}\cos(x^2)}{x}\right)\right|_{x=\sqrt{\pi}}$

* **62. (The continuity of sine and cosine)**

(a) Prove that

$$\lim_{\theta\to 0} \sin\theta = 0 \quad \text{and} \quad \lim_{\theta\to 0} \cos\theta = 1$$

as follows: use the fact that the length of chord AP is less than the length of arc AP in Figure 2.24 to show that

$$\sin^2\theta + (1 - \cos\theta)^2 < \theta^2.$$

Then deduce that $0 \le |\sin\theta| < |\theta|$ and $0 \le |1 - \cos\theta| < |\theta|$. Then use the Squeeze Theorem from Section 1.2.

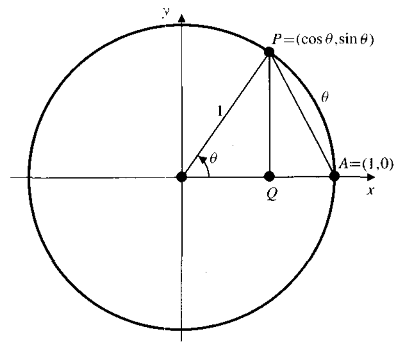

Figure 2.24

(b) Part (a) says that $\sin\theta$ and $\cos\theta$ are continuous at $\theta = 0$. Use the addition formulas to prove that they are therefore continuous at every θ.

2.6 The Mean-Value Theorem

If you set out in a car at 1:00 p.m. and arrive in a town 150 km away from your starting point at 3:00 p.m., then you have travelled at an average speed of $150/2 = 75$ km/h. Although you may not have travelled at constant speed, you must have been going 75 km/h at *at least one instant* during your journey, for if your speed was always less than 75 km/h you would have gone less than 150 km in 2 h, and if your speed was always more than 75 km/h, you would have gone more than 150 km in 2 h. In order to get from a value less than 75 km/h to a value greater than 75 km/h, your speed, which is a continuous function of time, must pass through the value 75 km/h at some intermediate time.

The conclusion that the average speed over a time interval must be equal to the instantaneous speed at some time in that interval is an instance of an important mathematical principle. In geometric terms it says that if A and B are two points on a smooth curve, then there is at least one point C on the curve between A and B where the tangent line is parallel to the chord line AB. See Figure 2.25.

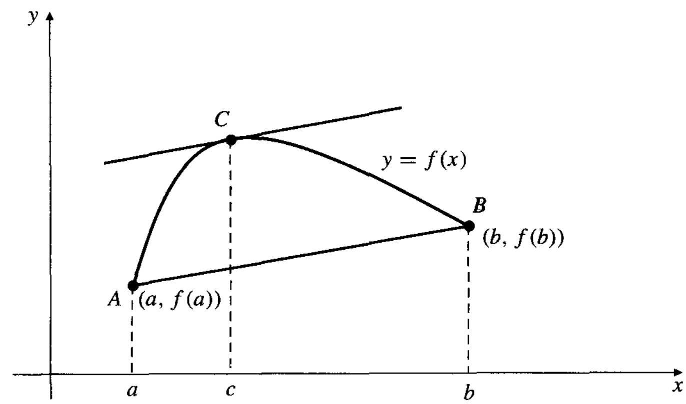

Figure 2.25 There is a point C on the curve where the tangent is parallel to the chord AB.

This principle is stated more precisely in the following theorem.

THEOREM 11

The Mean-Value Theorem

Suppose that the function f is continuous on the closed, finite interval $[a, b]$ and that it is differentiable on the open interval $]a, b[$. Then there exists a point c in the open interval $]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

This says that the slope of the chord line joining the points $(a, f(a))$ and $(b, f(b))$ is equal to the slope of the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$, so the two lines are parallel.

We will prove the Mean-Value Theorem later in this section. For now we make several observations.

1. The hypotheses of the Mean-Value Theorem are all necessary for the conclusion; if f fails to be continuous at even one point of $[a, b]$ or fails to be differentiable at even one point of $]a, b[$, then there may be no point where the tangent line is parallel to the secant line AB. See Figure 2.26.

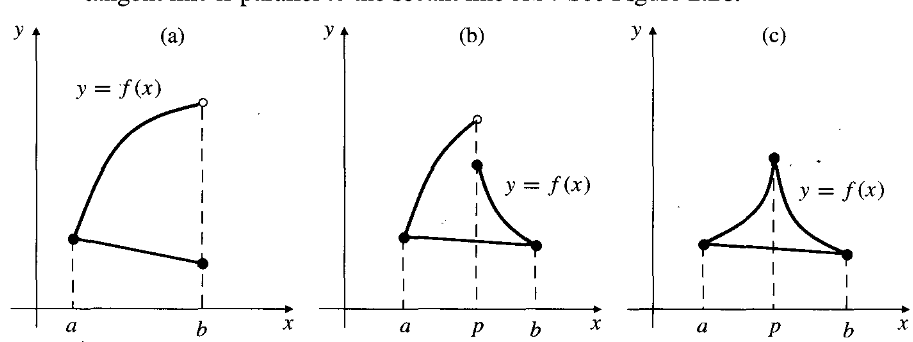

Figure 2.26 Functions that fail to satisfy the hypotheses of the Mean-Value Theorem, and for which the conclusion is false:

(a) f is discontinuous at endpoint b

(b) f is discontinuous at p

(c) f is not differentiable at p

2. The Mean-Value Theorem gives no indication of how many points C there may be on the curve between A and B where the tangent is parallel to AB. If the curve is itself the straight line AB, then every point on the line between A and B has the required property. In general, there may be more than one point (see Figure 2.27); the Mean-Value Theorem asserts only that there must be at least one.

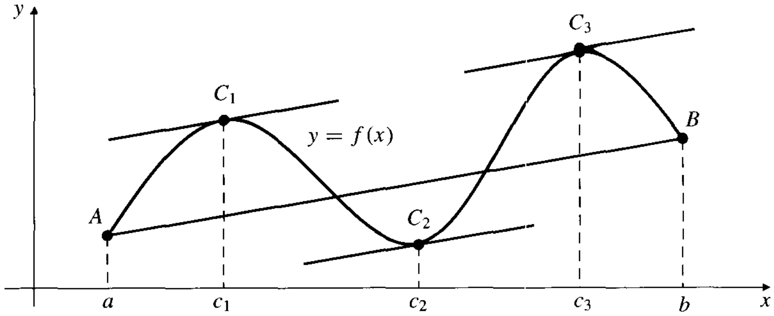

Figure 2.27 For this curve there are three points C where the tangent is parallel to the chord AB

3. The Mean-Value Theorem gives us no information on how to find the point c, which it says must exist. For some simple functions it is possible to calculate c (see the following example), but doing so is usually of no practical value. As we shall see, the importance of the Mean-Value Theorem lies in its use as a theoretical tool. It belongs to a class of theorems called *existence theorems*, as do the Max-Min Theorem and the Intermediate-Value Theorem (Theorems 8 and 9 of Section 1.4).

Example 1 Verify the conclusion of the Mean-Value Theorem for $f(x) = \sqrt{x}$ on the interval $[a, b]$, where $0 \le a < b$.

Solution The theorem says that there must be a number c in the interval $]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{2\sqrt{c}} = \frac{\sqrt{b} - \sqrt{a}}{b - a} = \frac{\sqrt{b} - \sqrt{a}}{(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})} = \frac{1}{\sqrt{b} + \sqrt{a}}.$$

Thus, $2\sqrt{c} = \sqrt{a} + \sqrt{b}$ and $c = \left(\frac{\sqrt{b} + \sqrt{a}}{2}\right)^2$. Since $a < b$ we have

$$a = \left(\frac{\sqrt{a} + \sqrt{a}}{2}\right)^2 < c < \left(\frac{\sqrt{b} + \sqrt{b}}{2}\right)^2 = b,$$

so c lies in the interval $]a, b[$. ■

The following two examples are more representative of how the Mean-Value Theorem is actually used.

Example 2 Show that $\sin x < x$ for all $x > 0$.

Solution If $x > 2\pi$, then $\sin x \le 1 < 2\pi < x$. If $0 < x \le 2\pi$, then, by the Mean-Value Theorem, there exists c in the open interval $]0, 2\pi[$ such that

$$\frac{\sin x}{x} = \frac{\sin x - \sin 0}{x - 0} = \frac{d}{dx}\sin x \bigg|_{x=c} = \cos c < 1.$$

Thus, $\sin x < x$ in this case too. ■

Example 3 Show that $\sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$ and for $-1 \le x < 0$.

Solution If $x > 0$, apply the Mean-Value Theorem to $f(x) = \sqrt{1+x}$ on the interval $[0, x]$. There exists c in $]0, x[$ such that

$$\frac{\sqrt{1+x}-1}{x} = \frac{f(x)-f(0)}{x-0} = f'(c) = \frac{1}{2\sqrt{1+c}} < \frac{1}{2}.$$

The last inequality holds because $c > 0$. Multiplying by the positive number x and transposing the -1 gives $\sqrt{1+x} < 1 + \frac{x}{2}$.

If $-1 \le x < 0$, we apply the Mean-Value Theorem to $f(x) = \sqrt{1+x}$ on the interval $[x, 0]$. There exists c in $]x, 0[$ such that

$$\frac{\sqrt{1+x}-1}{x} = \frac{1-\sqrt{1+x}}{-x} = \frac{f(0)-f(x)}{0-x} = f'(c) = \frac{1}{2\sqrt{1+c}} > \frac{1}{2}$$

(because $0 < 1 + c < 1$). Now we must multiply by the negative number x, which reverses the inequality, $\sqrt{1+x} - 1 < \frac{x}{2}$, and the required inequality again follows by transposing the -1. ■

Increasing and Decreasing Functions

Intervals on which the graph of a function f has positive or negative slope provide useful information about the behaviour of f. The Mean-Value Theorem enables us to determine such intervals by considering the sign of the derivative f'.

DEFINITION 5

Increasing and decreasing functions

Suppose that the function f is defined on an interval I and that x_1 and x_2 are two points of I.

(a) If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$ we say f is **increasing** on I.
(b) If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$ we say f is **decreasing** on I.
(c) If $f(x_2) \ge f(x_1)$ whenever $x_2 > x_1$ we say f is **nondecreasing** on I.
(d) If $f(x_2) \le f(x_1)$ whenever $x_2 > x_1$ we say f is **nonincreasing** on I.

Figure 2.28 illustrates these terms. Note the distinction between *increasing* and *nondecreasing*. If a function is increasing (or decreasing) on an interval, it must take different values at different points. (Such a function is called **one-to-one**.) A nondecreasing function (or a nonincreasing function) may be constant on all or part of an interval and may therefore not be one-to-one.

THEOREM 12

Let J be an open interval, and let I be an interval consisting of all the points in J and possibly one or both of the endpoints of J. Suppose that f is continuous on I and differentiable on J.

(a) If $f'(x) > 0$ for all x in J, then f is increasing on I.
(b) If $f'(x) < 0$ for all x in J, then f is decreasing on I.
(c) If $f'(x) \ge 0$ for all x in J, then f is nondecreasing on I.
(d) If $f'(x) \le 0$ for all x in J, then f is nonincreasing on I.

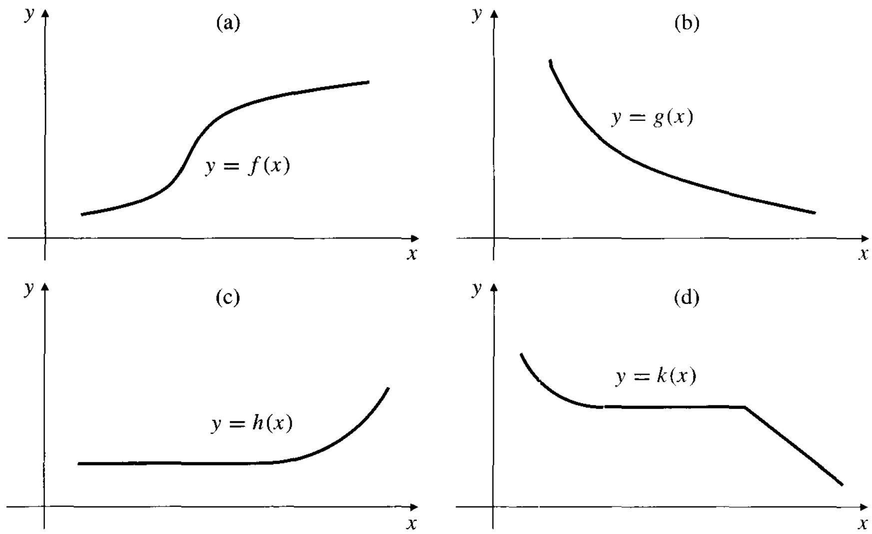

Figure 2.28
(a) Function f is increasing
(b) Function g is decreasing
(c) Function h is nondecreasing
(d) Function k is nonincreasing

PROOF Let x_1 and x_2 be points in I with $x_2 > x_1$. By the Mean-Value Theorem there exists a point c in $]x_1, x_2[$ (and therefore in J) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c);$$

hence, $f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$. Since $x_2 - x_1 > 0$, the difference $f(x_2) - f(x_1)$ has the same sign as $f'(c)$ and may be zero if $f'(c)$ is zero. Thus, all four conclusions follow from the corresponding parts of Definition 5.

Remark Despite Theorem 12, $f'(x_0) > 0$ at a single point x_0 does *not* imply that f is increasing on *any* interval containing x_0. See Exercise 18 at the end of this section for a counterexample.

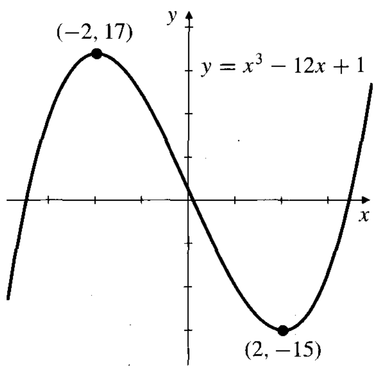

Figure 2.29

Example 4 On what intervals is the function $f(x) = x^3 - 12x + 1$ increasing? On what intervals is it decreasing?

Solution We have $f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$. Observe that $f'(x) > 0$ if $x < -2$ or $x > 2$ and $f'(x) < 0$ if $-2 < x < 2$. Therefore, f is increasing on the intervals $]-\infty, -2[$ and $]2, \infty[$ and is decreasing on the interval $]-2, 2[$. See Figure 2.29.

A function f whose derivative satisfies $f'(x) \geq 0$ on an interval can still be increasing there, rather than just nondecreasing as assured by Theorem 12(c). This will happen if $f'(x) = 0$ only at isolated points, so that f is assured to be increasing on intervals to the left and right of these points.

Example 5 Show that $f(x) = x^3$ is increasing on any interval.

Solution Let x_1 and x_2 be any two real numbers satsifying $x_1 < x_2$. Since $f'(x) = 3x^2 > 0$ except at $x = 0$, Theorem 12(a) tells us that $f(x_1) < f(x_2)$ if either $x_1 < x_2 \leq 0$ or $0 \leq x_1 < x_2$. If $x_1 < 0 < x_2$ then $f(x_1) < 0 < f(x_2)$. Thus, f is increasing on every interval.

If a function is constant on an interval, then its derivative is zero on that interval. The Mean-Value Theorem provides a converse of this fact.

THEOREM 13

If f is continuous on an interval I, and $f'(x) = 0$ at every interior point of I (i.e., at every point of I that is not an endpoint of I), then $f(x) = C$, a constant, on I.

PROOF Pick a point x_0 in I and let $C = f(x_0)$. If x is any other point of I, then the Mean-Value Theorem says that there exists a point c between x_0 and x such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(c).$$

The point c must belong to I because an interval contains all points between any two of its points, and c cannot be an endpoint of I since $c \neq x_0$ and $c \neq x$. Since $f'(c) = 0$ for all such points c, we have $f(x) - f(x_0) = 0$ for all x in I, and $f(x) = f(x_0) = C$ as claimed.

We will see how Theorem 13 can be used to establish identities for new functions encountered in later chapters. We will also use it when finding antiderivatives in Section 2.10.

Proof of the Mean-Value Theorem

The Mean-Value Theorem is one of those deep results that is based on the completeness of the real number system via the fact that a continuous function on a closed, finite interval takes on a maximum and minimum value (Theorem 8 of Section 1.4). Before giving the proof we establish two preliminary results.

THEOREM 14

If f is defined on an open interval $]a, b[$ and achieves a maximum (or minimum) value at the point c in $]a, b[$, and if $f'(c)$ exists, then $f'(c) = 0$. (Values of x where $f'(x) = 0$ are called **critical points** of the function f.)

PROOF Suppose that f has a maximum value at c. Then $f(x) - f(c) \leq 0$ whenever x is in $]a, b[$. If $c < x < b$, then

$$\frac{f(x) - f(c)}{x - c} \leq 0, \qquad \text{so} \quad f'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \leq 0.$$

Similarly, if $a < x < c$, then

$$\frac{f(x) - f(c)}{x - c} \geq 0, \qquad \text{so} \quad f'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Thus $f'(c) = 0$. The proof for a minimum value at c is similar.

THEOREM 15

Rolle's Theorem

Suppose that the function g is continuous on the closed, finite interval $[a, b]$ and that it is differentiable on the open interval $]a, b[$. If $g(a) = g(b)$, then there exists a point c in the open interval $]a, b[$ such that $g'(c) = 0$.

PROOF If $g(x) = g(a)$ for every x in $[a, b]$, then g is a constant function, so $g'(c) = 0$ for every c in $]a, b[$. Therefore, suppose there exists x_0 in $]a, b[$ such that $g(x_0) \neq g(a)$. Let us assume that $g(x_0) > g(a)$. (If $g(x_0) < g(a)$, the proof is similar.) By the Max-Min Theorem (Theorem 8 of Section 1.4), being continuous on $[a, b]$, g must have a maximum value at some point c in $[a, b]$. Since

$$g(c) \geq g(x_0) > g(a) = g(b),$$

c cannot be either a or b. Therefore, c is in the open interval $]a, b[$, so g is differentiable at c. By Theorem 14, c must be a critical point of g: $g'(c) = 0$.

Remark Rolle's Theorem is a special case of the Mean-Value Theorem in which the chord line has slope 0, so the corresponding parallel tangent line must also have slope 0. We can deduce the Mean-Value Theorem from this special case.

Proof of the Mean-Value Theorem Suppose f satisfies the conditions of the Mean-Value Theorem. Let

$$g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right).$$

(For $a \leq x \leq b$, $g(x)$ is the vertical displacement between the curve $y = f(x)$ and the chord line

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

joining $(a, f(a))$ and $(b, f(b))$. See Figure 2.30.)

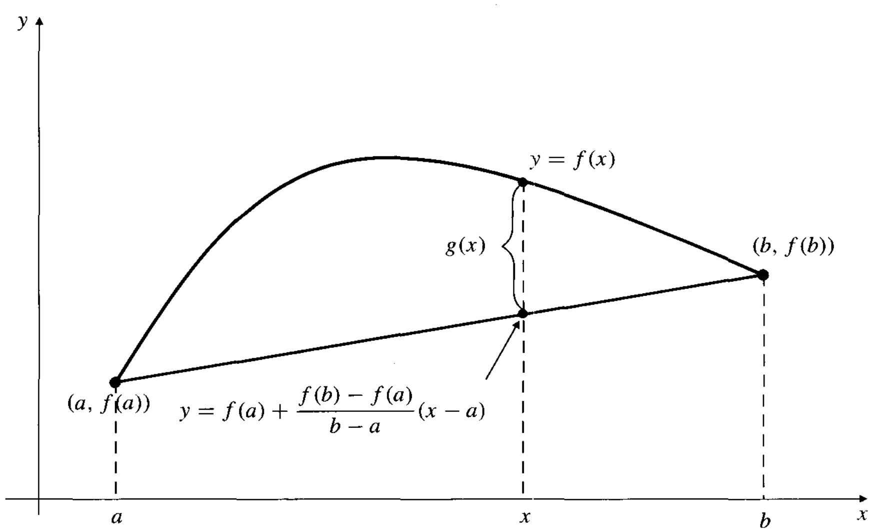

Figure 2.30 $g(x)$ is the vertical distance between the graph of f and the chord line

The function g is also continuous on $[a, b]$ and differentiable on $]a, b[$ because f has these properties. In addition, $g(a) = g(b) = 0$. By Rolle's Theorem, there is some point c in $]a, b[$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

it follows that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Many of the applications we will make of the Mean-Value Theorem in later chapters will actually use the following generalized version of it.

THEOREM 16

The Generalized Mean-Value Theorem

If functions f and g are both continuous on $[a, b]$ and differentiable on $]a, b[$, and if $g'(x) \neq 0$ for every x in $]a, b[$, then there exists a number c in $]a, b[$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

PROOF Note that $g(b) \neq g(a)$; otherwise, there would be some number in $]a, b[$ where $g' = 0$. Hence, neither denominator above can be zero. Apply the Mean-Value Theorem to

$$h(x) = \big(f(b) - f(a)\big)\big(g(x) - g(a)\big) - \big(g(b) - g(a)\big)\big(f(x) - f(a)\big).$$

Since $h(a) = h(b) = 0$, there exists c in $]a, b[$ such that $h'(c) = 0$. Thus,

$$\big(f(b) - f(a)\big)g'(c) - \big(g(b) - g(a)\big)f'(c) = 0,$$

and the result follows on division by the g factors.

Exercises 2.6

In Exercises 1–3, illustrate the Mean-Value Theorem by finding any points in the open interval $]a, b[$ where the tangent line to $y = f(x)$ is parallel to the chord line joining $(a, f(a))$ and $(b, f(b))$.

1. $f(x) = x^2$ on $[a, b]$

2. $f(x) = \frac{1}{x}$ on $[1, 2]$

3. $f(x) = x^3 - 3x + 1$ on $[-2, 2]$

* **4.** By applying the Mean-Value Theorem to $f(x) = \cos x + \frac{x^2}{2}$ on the interval $[0, x]$, and using the result of Example 2, show that

$$\cos x > 1 - \frac{x^2}{2}$$

for $x > 0$. This inequality is also true for $x < 0$. Why?

5. Show that $\tan x > x$ for $0 < x < \pi/2$.

6. Let $r > 1$. If $x > 0$ or $-1 \leq x < 0$, show that $(1 + x)^r > 1 + rx$.

7. Let $0 < r < 1$. If $x > 0$ or $-1 \leq x < 0$, show that $(1 + x)^r < 1 + rx$.

Find the intervals of increase and decrease of the functions in Exercises 8–15.

8. $f(x) = x^2 + 2x + 2$

9. $f(x) = x^3 - 4x + 1$

10. $f(x) = x^3 + 4x + 1$

11. $f(x) = (x^2 - 4)^2$

12. $f(x) = \frac{1}{x^2 + 1}$

13. $f(x) = x^3(5 - x)^2$

14. $f(x) = x - 2\sin x$

15. $f(x) = x + \sin x$

* **16.** If $f(x)$ is differentiable on an interval I and vanishes at $n \geq 2$ distinct points of I, prove that $f'(x)$ must vanish at at least $n - 1$ points in I.

17. What is wrong with the following "proof" of the Generalized Mean-Value Theorem? By the Mean-Value Theorem, $f(b) - f(a) = (b - a)f'(c)$ for some c between a and b and, similarly, $g(b) - g(a) = (b - a)g'(c)$ for some such c. Hence, $(f(b) - f(a))/(g(b) - g(a)) = f'(c)/g'(c)$, as required.

* **18.** Let $f(x) = \begin{cases} x + 2x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(a) Show that $f'(0) = 1$. (*Hint:* use the definition of derivative.)

(b) Show that any interval containing $x = 0$ also contains points where $f'(x) < 0$, so f cannot be increasing on such an interval.

2.7 Using Derivatives

In this section we will look at some examples of ways in which derivatives are used to represent and interpret changes and rates of change in the world around us. It is natural to think of change in terms of dependence on time, such as the velocity of a moving object, but there is no need to be so restrictive. Change with respect to variables other than time can be treated in the same way. For example, a physician may want to know how small changes in dosage can affect the body's response to a drug. An economist may want to study how foreign investment changes with respect to variations in a country's interest rates. These questions can all be formulated in terms of rate of change of a function with respect to a variable.

Approximating Small Changes

If one quantity, say y, is a function of another quantity x, that is

$$y = f(x),$$

we sometimes want to know how a change in the value of x by an amount Δx will affect the value of y. The exact change Δy in y is given by

$$\Delta y = f(x + \Delta x) - f(x),$$

but if the change Δx is small, then we can get a good approximation to Δy by using the fact that $\Delta y/\Delta x$ is approximately the derivative dy/dx. Thus

$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x \approx \frac{dy}{dx}\Delta x = f'(x)\,\Delta x.$$

Sometimes changes in a quantity are measured with respect to the size of the quantity. The **relative change** in x is the ratio $\Delta x/x$; the **percentage change** in x is the relative change expressed as a percentage:

$$\textbf{relative change in } x = \frac{\Delta x}{x}$$

$$\textbf{percentage change in } x = 100\,\frac{\Delta x}{x}.$$

Example 1 By approximately what percentage does the area of a circle increase if the radius increases by 2%?

Solution The area A of a circle is given in terms of the radius r by $A = \pi r^2$. Thus,

$$\Delta A \approx \frac{dA}{dr}\,\Delta r = 2\pi r\,\Delta r.$$

We divide this approximation by $A = \pi r^2$ to get an approximation that links the relative changes in A and r:

$$\frac{\Delta A}{A} \approx \frac{2\pi r\,\Delta r}{\pi r^2} = 2\,\frac{\Delta r}{r}.$$

If r increases by 2%, then $\Delta r = \frac{2}{100}\,r$, so

$$\frac{\Delta A}{A} \approx 2 \times \frac{2}{100} = \frac{4}{100}.$$

Thus, A increases by approximately 4%. ■

Average and Instantaneous Rates of Change

Recall the concept of average rate of change of a function over an interval, introduced in Section 1.1. The derivative of the function is the limit of this average rate as the length of the interval goes to zero, and so represents the rate of change of the function at a given value of its variable.

DEFINITION 6

The **average rate of change** of a function $f(x)$ with respect to x over the interval from a to $a + h$ is

$$\frac{f(a+h) - f(a)}{h}.$$

The **(instantaneous) rate of change** of f with respect to x at $x = a$ is the derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

It is conventional to use the word *instantaneous* even when x does not represent time, although the word is frequently omitted. When we say *rate of change* we mean *instantaneous rate of change*.

Example 2 How fast is area A of a circle increasing with respect to its radius when the radius is 5 m?

Solution The rate of change of the area with respect to the radius is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r.$$

When $r = 5$ m, the area is changing at the rate $2\pi \times 5 = 10\pi$ m^2/m. This means that a small change Δr m in the radius when the radius is 5 m would result in a change of about $10\pi \Delta r$ m^2 in the area of the circle.

The above example suggests that the appropriate units for the rate of change of a quantity y with respect to another quantity x are units of y per unit of x.

If $f'(x_0) = 0$, we say that f is **stationary** at x_0 and call x_0 a **critical point** of f. The corresponding point $(x_0, f(x_0))$ on the graph of f is also called a **critical point** of the graph. The graph has a horizontal tangent at a critical point, and f may or may not have a maximum or minimum value there. (See Figure 2.31.) It is still possible for f to be increasing or decreasing on an open interval containing a critical point. (See point a in Figure 2.31.)

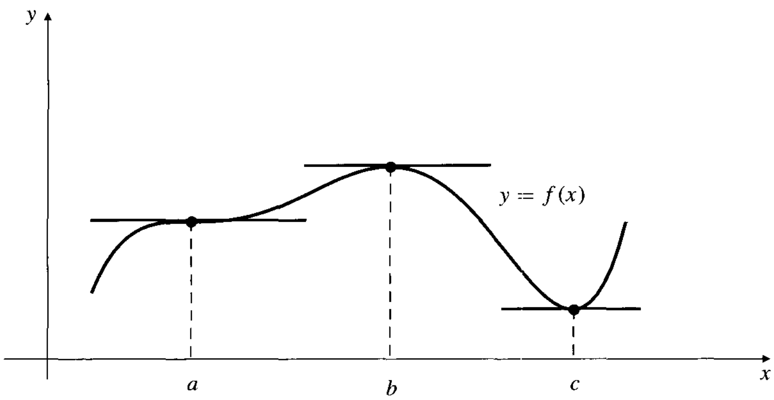

Figure 2.31 Critical points of f

Example 3 Suppose the temperature at a certain location t hours after noon on a certain day is T°C (T degrees Celsius), where

$$T = \frac{1}{3}t^3 - 3t^2 + 8t + 10 \qquad (\text{for } 0 \le t \le 5).$$

How fast is the temperature rising or falling at 1:00 p.m.? at 3:00 p.m.? At what instants is the temperature stationary?

Solution The rate of change of the temperature is given by

$$\frac{dT}{dt} = t^2 - 6t + 8 = (t-2)(t-4).$$

If $t = 1$, then $\dfrac{dT}{dt} = 3$, so the temperature is rising at rate 3°C/h at 1:00 p.m.

If $t = 3$, then $\dfrac{dT}{dt} = -1$, so the temperature is falling at a rate of 1°C/h at 3:00 p.m.

The temperature is stationary when $\dfrac{dT}{dt} = 0$, that is, at 2:00 p.m. and 4:00 p.m. ■

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is very **sensitive** to changes in x. The derivative $f'(x)$ is a measure of the sensitivity of the dependence of f on x.

Example 4 **(Dosage of a medicine)** A pharmacologist studying a drug that has been developed to lower blood pressure determines experimentally that the average reduction R in blood pressure resulting from a daily dosage of x mg of the drug is given by

$$R = 24.2\left(1 + \frac{x-13}{\sqrt{x^2 - 26x + 529}}\right) \quad \text{mm Hg.}$$

(The units are millimetres of mercury (Hg).) Determine the sensitivity of R to dosage x at dosage levels of 5 mg, 15 mg, and 35 mg. At which of these dosage levels would an increase in the dosage have the greatest effect?

Solution The sensitivity of R to x is dR/dx. We have

$$\begin{aligned}\frac{dR}{dx} &= 24.2\left(\frac{\sqrt{x^2-26x+529}(1) - (x-13)\dfrac{x-13}{\sqrt{x^2-26x+529}}}{x^2-26x+529}\right)\\ &= 24.2\left(\frac{x^2-26x+529-(x^2-26x+169)}{(x^2-26x+529)^{3/2}}\right)\\ &= \frac{8712}{(x^2-26x+529)^{3/2}}.\end{aligned}$$

At dosages $x = 5$ mg, 15 mg, and 35 mg we have sensitivities of

$$\left.\frac{dR}{dx}\right|_{x=5} = 0.998 \text{ mm Hg/mg}, \qquad \left.\frac{dR}{dx}\right|_{x=15} = 1.254 \text{ mm Hg/mg},$$

$$\left.\frac{dR}{dx}\right|_{x=35} = 0.355 \text{ mm Hg/mg}.$$

Among these three levels, the greatest sensitivity is at 15 mg. Increasing the dosage from 15 to 16 mg/day could be expected to further reduce average blood pressure by about 1.25 mm Hg. ■

Derivatives in Economics

Just as physicists use terms such as *velocity* and *acceleration* to refer to derivatives of certain quantities, economists also have their own specialized vocabulary for derivatives. They call them marginals. In economics the term **marginal** denotes the rate of change of a quantity with respect to a variable on which it depends. For example, the **cost of production** $C(x)$ in a manufacturing operation is a function of x, the number of units of product produced. The **marginal cost of production** is the rate of change of C with respect to x, so it is dC/dx. Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one more unit, that is,

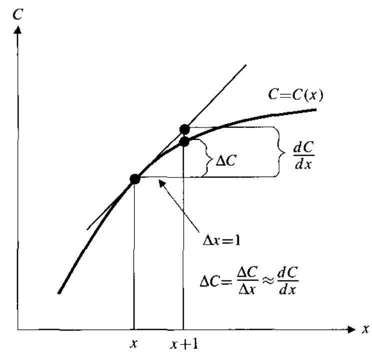

Figure 2.32 The marginal cost dC/dx is approximately the extra cost ΔC of producing $\Delta x = 1$ more unit

$$\Delta C = C(x + 1) - C(x).$$

To see why this is approximately correct, observe from Figure 2.32 that if the slope of $C = C(x)$ does not change quickly near x, then the difference quotient $\Delta C/\Delta x$ will be close to its limit, the derivative dC/dx, even if $\Delta x = 1$.

Example 5 **(Marginal Tax Rates)** If your marginal income tax rate is 35% and your income increases by \$1,000, you can expect to have to pay an extra \$350 in income taxes. This does not mean that you pay 35% of your entire income in taxes. It just means that at your current income level I, the rate of increase of taxes T with respect to income is $dT/dI = 0.35$. You will pay \$0.35 out of every extra dollar you earn in taxes. Of course, if your income increases greatly, you may land in a higher tax bracket and your marginal rate will increase. ■

Example 6 **(Marginal Cost of Production)** The cost of producing x tons of coal per day in a mine is \$$C(x)$ where

$$C(x) = 4{,}200 + 5.40x - 0.001x^2 + 0.000002x^3.$$

(a) What is the average cost of producing each ton if the daily production level is 1,000 tons? 2,000 tons?

(b) Find the marginal cost of production if the daily production level is 1,000 tons. 2,000 tons.

(c) If the production level increases slightly from 1,000 tons or from 2,000 tons, what will happen to the average cost per ton?

Solution

(a) The average cost per ton of coal is

$$\frac{C(x)}{x} = \frac{4,200}{x} + 5.40 - 0.001x + 0.000002x^2.$$

If $x = 1{,}000$, the average cost per ton is $C(1{,}000)/1{,}000 = \$10.6$ /ton. If $x = 2{,}000$, the average cost per ton is $C(2{,}000)/2{,}000 = \$13.5$ /ton.

(b) The marginal cost of production is

$$C'(x) = 5.40 - 0.002x + 0.000006x^2.$$

If $x = 1{,}000$, the marginal cost is $C'(1{,}000) = \$9.4$ /ton. If $x = 2{,}000$, the marginal cost is $C'(2{,}000) = \$25.4$ /ton.

(c) If the production level x is increased slightly from $x = 1{,}000$, then the average cost per ton will drop because the cost is increasing at a rate lower than the average cost. At $x = 2{,}000$ the opposite is true; an increase in production will increase the average cost per ton.

Economists sometimes prefer to measure relative rates of change that do not depend on the units used to measure the quantities involved. They use the term **elasticity** for such relative rates.

Example 7 **(Elasticity of demand)** The demand y for a certain product (i.e., the amount that can be sold) typically depends on the price p charged for the product: $y = f(p)$. The marginal demand $dy/dp = f'(p)$ (which is typically negative) depends on the units used to measure y and p. The *elasticity of the demand* is the quantity

$$-\frac{p}{y}\frac{dy}{dp} \qquad \text{(the "}-\text{" sign ensures elasticity is positive),}$$

which is independent of units and provides a good measure of the sensitivity of demand to changes in price. To see this, suppose that new units of demand and price are introduced, which are multiples of the old units. In terms of the new units the demand and price are now Y and P, where

$$Y = k_1 y \qquad \text{and} \qquad P = k_2 p.$$

Thus, $Y = k_1 f(P/k_2)$ and $dY/dP = (k_1/k_2)f'(P/k_2) = (k_1/k_2)f'(p)$ by the Chain Rule. It follows that the elasticity has the same value:

$$-\frac{P}{Y}\frac{dY}{dP} = -\frac{k_2 p}{k_1 y}\frac{k_1}{k_2}f'(p) = -\frac{p}{y}\frac{dy}{dp}.$$

Exercises 2.7

In Exercises 1–6, find the approximate percentage changes in the given function $y = f(x)$ that will result from an increase of 2% in the value of x.

1. $y = x^2$
2. $y = 1/x$
3. $y = 1/x^2$
4. $y = x^3$
5. $y = \sqrt{x}$
6. $y = x^{-2/3}$
7. By approximately what percentage will the volume $(V = \frac{4}{3}\pi r^3)$ of a ball of radius r increase if the radius increases by 2%?
8. By about what percentage will the edge length of an ice cube decrease if the cube loses 6% of its volume by melting?
9. Find the rate of change of the area of a square with respect to the length of its side when the side is 4 ft.
10. Find the rate of change of the side of a square with respect to the area of the square when the area is 16 m^2.
11. Find the rate of change of the diameter of a circle with respect to its area.
12. Find the rate of change of the area of a circle with respect to its diameter.
13. Find the rate of change of the volume of a sphere (given by $V = \frac{4}{3}\pi r^3$) with respect to its radius r when the radius is 2 m.

14. What is the rate of change of the area A of a square with respect to the length L of the diagonal of the square?

15. What is the rate of change of the circumference C of a circle with respect to the area A of the circle?

16. Find the rate of change of the side s of a cube with respect to the volume V of the cube.

What are the critical points of the functions in Exercises 17–20? On what intervals is each function increasing and decreasing?

17. $f(x) = x^2 - 4$ **18.** $f(x) = x^3 - 12x + 1$

19. $y = x^3 + 6x^2$ **20.** $y = 1 - x - x^5$

21. Show that $f(x) = x^3$ is increasing on the whole real line even though $f'(x)$ is not positive at every point.

22. On what intervals is $f(x) = x + 2\sin x$ increasing?

Use a graphing utility or a computer algebra system to find the critical points of the functions in Exercises 23–26 correct to 6 decimal places.

23. $f(x) = \dfrac{x^2 - x}{x^2 - 4}$ **24.** $f(x) = \dfrac{2x + 1}{x^2 + x + 1}$

25. $f(x) = x - \sin\left(\dfrac{x}{x^2 + x + 1}\right)$

26. $f(x) = \dfrac{\sqrt{1 - x^2}}{\cos(x + 0.1)}$

27. The volume of water in a tank t min after it starts draining is

$$V(t) = 350(20 - t)^2 \text{ L}.$$

(a) How fast is the water draining out after 5 min? after 15 min?

(b) What is the average rate at which water is draining out during the time interval from 5 to 15 min?

28. **(Poiseuille's Law)** The flow rate F (in litres per minute) of a liquid through a pipe is proportional to the fourth power of the radius of the pipe:

$$F = kr^4.$$

Approximately what percentage increase is needed in the radius of the pipe to increase the flow rate by 10%?

29. **(Gravitational force)** The gravitational force F with which the earth attracts an object in space is given by $F = k/r^2$, where k is a constant and r is the distance from the object to the centre of the earth. If F decreases with respect to r at rate 1 *pound/mile* when $r = 4{,}000$ mi, how fast does F change with respect to r when $r = 8{,}000$ mi?

30. **(Sensitivity of revenue to price)** The sales revenue $\$R$ from a software product depends on the price $\$p$ charged by the distributor according to the formula

$$R = 4{,}000p - 10p^2.$$

(a) How sensitive is R to p when $p = \$100$? $p = \$200$? $p = \$300$?

(b) Which of these three is the most reasonable price for the distributor to charge? Why?

31. **(Marginal cost)** The cost of manufacturing x refrigerators is $\$C(x)$, where

$$C(x) = 8{,}000 + 400x - 0.5x^2.$$

(a) Find the marginal cost if 100 refrigerators are manufactured.

(b) Show that the marginal cost is approximately the difference in cost of manufacturing 101 refrigerators instead of 100.

32. **(Marginal profit)** If a plywood factory produces x sheets of plywood per day, its profit per day will be $\$P(x)$, where

$$P(x) = 8x - 0.005x^2 - 1{,}000.$$

(a) Find the marginal profit. For what values of x is the marginal profit positive? negative?

(b) How many sheets should be produced each day to generate maximum profits?

33. The cost C (in dollars) of producing n widgets per month in a widget factory is given by

$$C = \frac{80{,}000}{n} + 4n + \frac{n^2}{100}.$$

Find the marginal cost of production if the number of widgets manufactured each month is (a) 100 and (b) 300.

* **34.** In a mining operation the cost C (in dollars) of extracting each tonne of ore is given by

$$C = 10 + \frac{20}{x} + \frac{x}{1{,}000},$$

where x is the number of tonnes extracted each day. (For small x, C decreases as x increases because of economies of scale, but for large x, C increases with x because of overloaded equipment and labour overtime.) If each tonne of ore can be sold for \$13, how many tonnes should be extracted each day to maximize the daily profit of the mine?

* **35.** **(Average cost and marginal cost)** If it costs a manufacturer $C(x)$ dollars to produce x items, then his average cost of production is $C(x)/x$ dollars per item. Typically the average cost is a decreasing function of x for small x and an increasing function of x for large x. (Why?) Show that the value of x that minimizes the average cost makes the average cost equal to the marginal cost.

36. **(Constant elasticity)** Show that if demand y is related to price p by the equation $y = Cp^{-r}$, where C and r are positive constants, then the elasticity of demand (see Example 7) is the constant r.

2.8 Higher-Order Derivatives

If the derivative $y' = f'(x)$ of a function $y = f(x)$ is itself differentiable at x, we can calculate *its* derivative, which we call the **second derivative** of f and denote by $y'' = f''(x)$. As is the case for first derivatives, second derivatives can be denoted by various notations depending on the context. Some of the more common ones are

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = D_x^2 y = D_x^2 f(x).$$

Similarly, you can consider third-, fourth-, and in general nth-order derivatives. The prime notation is inconvenient for derivatives of high order, so we denote the order by a superscript in parentheses (to distinguish it from an exponent): the nth derivative of $y = f(x)$ is

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}f(x) = D_x^n y = D_x^n f(x),$$

and it is defined to be the derivative of the $(n-1)$st derivative. For $n = 1$, 2, and 3, primes are still normally used: $f^{(2)}(x) = f''(x)$, $f^{(3)}(x) = f'''(x)$. It is sometimes convenient to denote $f^{(0)}(x) = f(x)$, that is, to regard a function as its own zeroth-order derivative.

Example 1 The **velocity** of a moving object is the (instantaneous) rate of change of the position of the object with respect to time; if the object moves along the x-axis and is at position $x = f(t)$ at time t, then its velocity at that time is

$$v = \frac{dx}{dt} = f'(t).$$

Similarly, the **acceleration** of the object is the rate of change of the velocity. Thus, the acceleration is the *second derivative* of the position:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t).$$

We will investigate the relationships between position, velocity, and acceleration further in Section 2.11.

Example 2 If $y = x^3$, then $y' = 3x^2$, $y'' = 6x$, $y''' = 6$, $y^{(4)} = 0$, and all higher derivatives are zero.

In general, if $f(x) = x^n$ (where n is a positive integer), then

$$\begin{aligned} f^{(k)}(x) &= n(n-1)(n-2)\ldots(n-(k-1))\,x^{n-k} \\ &= \begin{cases} \dfrac{n!}{(n-k)!}x^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n, \end{cases} \end{aligned}$$

where $n!$ (called n **factorial**) is defined by:

$$\begin{aligned}
0! &= 1\\
1! &= 0! \times 1 = 1 \times 1 = 1\\
2! &= 1! \times 2 = 1 \times 2 = 2\\
3! &= 2! \times 3 = 1 \times 2 \times 3 = 6\\
4! &= 3! \times 4 = 1 \times 2 \times 3 \times 4 = 24\\
&\vdots\\
n! &= (n-1)! \times n = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.
\end{aligned}$$

It follows that if P is a polynomial of degree n,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n,\ a_{n-1},\ \ldots,\ a_1,\ a_0$ are constants, then $P^{(k)}(x) = 0$ for $k > n$. For $k \le n$, $P^{(k)}$ is a polynomial of degree $n - k$; in particular, $P^{(n)}(x) = n!\,a_n$, a constant function.

Example 3 Show that if A, B, and k are constants, then the function

$$y = A\cos(kt) + B\sin(kt)$$

is a solution of the *second-order* **differential equation of simple harmonic motion** (see Section 3.7):

$$\frac{d^2y}{dt^2} + k^2 y = 0.$$

Solution To be a solution, the function $y(t)$ must satisfy the differential equation *identically*; that is,

$$\frac{d^2}{dt^2}y(t) + k^2 y(t) = 0$$

must hold for every real number t. We verify this by calculating the first two derivatives of the given function $y(t) = A\cos(kt) + B\sin(kt)$ and observing that the second derivative plus $k^2 y(t)$ is, in fact, zero everywhere:

$$\begin{aligned}
\frac{dy}{dt} &= -Ak\sin(kt) + Bk\cos(kt)\\
\frac{d^2y}{dt^2} &= -Ak^2\cos(kt) - Bk^2\sin(kt) = -k^2 y(t),\\
\frac{d^2y}{dt^2} &+ k^2 y(t) = 0.
\end{aligned}$$

■

Example 4 Find the nth derivative, $y^{(n)}$, of $y = \dfrac{1}{1+x} = (1+x)^{-1}$.

Solution Begin by calculating the first few derivatives:

$$\begin{aligned}
y' &= -(1+x)^{-2}\\
y'' &= -(-2)(1+x)^{-3} = 2(1+x)^{-3}\\
y''' &= 2(-3)(1+x)^{-4} = -3!(1+x)^{-4}\\
y^{(4)} &= -3!(-4)(1+x)^{-5} = 4!(1+x)^{-5}
\end{aligned}$$

The pattern here is becoming obvious. It seems that

$$y^{(n)} = (-1)^n n!(1+x)^{-n-1}.$$

Note the use of $(-1)^n$ to denote a positive sign if n is even and a negative sign if n is odd.

We have not yet actually proved that the above formula is correct for every n, although it is clearly correct for $n = 1, 2, 3$, and 4. To complete the proof we use mathematical induction (Section 2.3). Suppose that the formula is valid for $n = k$, where k is some positive integer. Consider $y^{(k+1)}$:

$$\begin{aligned}
y^{(k+1)} &= \frac{d}{dx}y^{(k)} = \frac{d}{dx}\Big((-1)^k k!(1+x)^{-k-1}\Big)\\
&= (-1)^k k!(-k-1)(1+x)^{-k-2} = (-1)^{k+1}(k+1)!(1+x)^{-(k+1)-1}.
\end{aligned}$$

This is what the formula predicts for the $(k+1)$st derivative. Therefore, if the formula for $y^{(n)}$ is correct for $n = k$, then it is also correct for $n = k+1$. Since the formula is known to be true for $n = 1$, it must therefore be true for every integer $n \ge 1$ *by induction.*

■

Example 5 Find a formula for $f^{(n)}(x)$, given that $f(x) = \sin(ax+b)$.

Solution Begin by calculating several derivatives:

$$\begin{aligned}
f'(x) &= a\cos(ax+b)\\
f''(x) &= -a^2\sin(ax+b) = -a^2 f(x)\\
f'''(x) &= -a^3\cos(ax+b) = -a^2 f'(x)\\
f^{(4)}(x) &= a^4\sin(ax+b) = a^4 f(x)\\
f^{(5)}(x) &= a^5\cos(ax+b) = a^4 f'(x)\\
&\ \vdots
\end{aligned}$$

The pattern is pretty obvious here. Each new derivative is $-a^2$ times the second previous one. A formula that gives all the derivatives is

$$f^{(n)}(x) = \begin{cases} (-1)^k a^n \sin(ax+b) & \text{if } n = 2k\\ (-1)^k a^n \cos(ax+b) & \text{if } n = 2k+1 \end{cases} \qquad (k = 0, 1, 2, \ldots),$$

which can also be verified by induction.

■

Our final example shows that it is not always easy to obtain a formula for the nth derivative of a function.

Example 6 Calculate f', f'', and f''' for $f(x) = \sqrt{x^2+1}$. Can you see enough of a pattern to predict $f^{(4)}$?

Solution Since $f(x) = (x^2+1)^{1/2}$ we have

$$\begin{aligned}
f'(x) &= \tfrac{1}{2}(x^2+1)^{-1/2}(2x) = x(x^2+1)^{-1/2},\\
f''(x) &= (x^2+1)^{-1/2} + x\left(-\tfrac{1}{2}\right)(x^2+1)^{-3/2}(2x)\\
&= (x^2+1)^{-3/2}(x^2+1-x^2) = (x^2+1)^{-3/2},\\
f'''(x) &= -\tfrac{3}{2}(x^2+1)^{-5/2}(2x) = -3x(x^2+1)^{-5/2}.
\end{aligned}$$

Although the expression obtained from each differentiation simplified somewhat, the pattern of these derivatives is not (yet) obvious enough to enable us to predict the formula for $f^{(4)}(x)$ without having to calculate it. In fact,

$$f^{(4)}(x) = 3(4x^2-1)(x^2+1)^{-7/2},$$

so the pattern (if there is one) doesn't become any clearer at this stage. ■

Remark Higher-order derivatives can be indicated in Maple by repeating the variable of differentiation or indicating the order by using the `$` operator:

```
> diff(x^5,x,x) + diff(sin(2*x),x$3);
```

$$20\,x^3 - 8\cos(2x)$$

The `D` operator can also be used for higher-order derivatives of a function (as distinct from an expression) by composing it explicitly or using the `@@` operator:

```
> f := x -> x^5; fpp := D(D(f)); (D@@3)(f)(a);
```

$$f := x \to x^5$$

$$fpp := x \to 20\,x^3$$

$$60\,a^2$$

Exercises 2.8

Find y', y'', and y''' for the functions in Exercises 1–12.

1. $y = (3-2x)^7$
2. $y = x^2 - \dfrac{1}{x}$
3. $y = \dfrac{6}{(x-1)^2}$
4. $y = \sqrt{ax+b}$
5. $y = x^{1/3} - x^{-1/3}$
6. $y = x^{10} + 2x^8$
7. $y = (x^2+3)\sqrt{x}$
8. $y = \dfrac{x-1}{x+1}$
9. $y = \tan x$
10. $y = \sec x$
11. $y = \cos(x^2)$
12. $y = \dfrac{\sin x}{x}$

In Exercises 13–23, calculate enough derivatives of the given function to enable you to guess the general formula for $f^{(n)}(x)$. Then verify your guess using mathematical induction.

13. $f(x) = \dfrac{1}{x}$
14. $f(x) = \dfrac{1}{x^2}$
15. $f(x) = \dfrac{1}{2-x}$
16. $f(x) = \sqrt{x}$
17. $f(x) = \dfrac{1}{a+bx}$
18. $f(x) = x^{2/3}$
19. $f(x) = \cos(ax)$
20. $f(x) = x\cos x$
21. $f(x) = x\sin(ax)$

*22. $f(x) = \dfrac{1}{|x|}$

*23. $f(x) = \sqrt{1-3x}$

24. If $y = \tan kx$, show that $y'' = 2k^2y(1+y^2)$.
25. If $y = \sec kx$, show that $y'' = k^2y(2y^2-1)$.

26. Use mathematical induction to prove that the nth derivative of $y = \sin(ax + b)$ is given by the formula asserted at the end of Example 5.

27. Use mathematical induction to prove that the nth derivative of $y = \tan x$ is of the form $P_{n+1}(\tan x)$, where P_{n+1} is a polynomial of degree $n + 1$.

28. If f and g are twice-differentiable functions, show that $(fg)'' = f''g + 2f'g' + fg''$.

* **29.** State and prove the results analogous to that of Exercise 28 but for $(fg)^{(3)}$ and $(fg)^{(4)}$. Can you guess the formula for $(fg)^{(n)}$?

* **30.** If $f''(x)$ exists on an interval I and if f vanishes at at least three distinct points of I, prove that f'' must vanish at some point in I.

* **31.** Generalize Exercise 30 to a function for which $f^{(n)}$ exists on I and for which f vanishes at at least $n + 1$ distinct points in I.

* **32.** Suppose f is twice differentiable on an interval I (i.e., f'' exists on I). Suppose that the points 0 and 2 belong to I and that $f(0) = f(1) = 0$ and $f(2) = 1$. Prove that:

(a) $f'(a) = \frac{1}{2}$ for some point a in I.

(b) $f''(b) > \frac{1}{2}$ for some point b in I.

(c) $f'(c) = \frac{1}{7}$ for some point c in I.

2.9 Implicit Differentiation

We know how to find the slope of a curve which is the graph of a function $y = f(x)$ by calculating the derivative of f. But not all curves are the graphs of such functions. To be the graph of a function $f(x)$, the curve must not intersect any vertical lines at more than one point.

Curves are generally the graphs of *equations* in two variables. Such equations can be written in the form

$$F(x, y) = 0,$$

where $F(x, y)$ denotes an expression involving the two variables x and y. For example, a circle with centre at the origin and radius 5 has equation

$$x^2 + y^2 - 25 = 0,$$

so $F(x, y) = x^2 + y^2 - 25$ for that circle.

Sometimes we can solve an equation $F(x, y) = 0$ for y and so find explicit formulas for one or more functions $y = f(x)$ defined by the equation. Usually, however, we are not able to solve the equation. However, we can still regard it as defining y as one or more functions of x *implicitly*, even it we cannot solve for these functions *explicitly*. Moreover, we still find the derivative dy/dx of these implicit solutions by a technique called **implicit differentiation**. The idea is to differentiate the given equation with respect to x, regarding y as a function of x having derivative dy/dx, or y'.

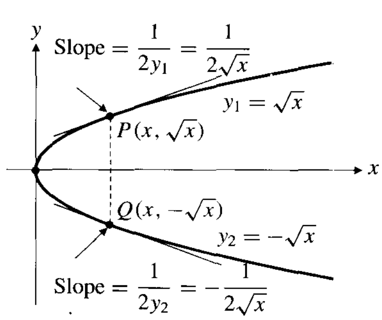

Figure 2.33 The equation $y^2 = x$ defines two differentiable functions of x on the interval $x \ge 0$

Example 1 Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x; in this case we know them explicitly. They are $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 2.33), having derivatives defined for $x > 0$ by

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

However, we can find the slope of the curve $y^2 = x$ at any point (x, y) satisfying that equation without first solving the equation for y. To find dy/dx we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating y as a differentiable function of x and using the Chain Rule to differentiate y^2:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x) \qquad \left(\text{The Chain Rule gives } \frac{d}{dx}y^2 = 2y\frac{dy}{dx}.\right)$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Observe that this agrees with the derivatives we calculated above for *both* of the explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

Example 2 Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x. Again, it combines the graphs of two functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 2.34). The point $(3, -4)$ lies on the graph of y_2, so we can find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

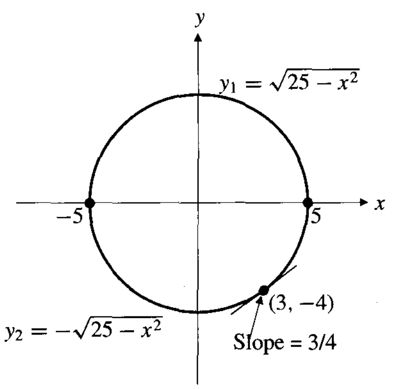

Figure 2.34 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$

But we can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $\left.-\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$

Example 3 Find $\frac{dy}{dx}$ if $y \sin x = x^3 + \cos y$.

Solution This time we cannot solve the equation for y as an explicit function of x, so we *must* use implicit differentiation.

To find dy/dx by implicit differentiation:

1. Differentiate both sides of the equation with respect to x, regarding y as a function of x and using the Chain Rule to differentiate functions of y.
2. Collect terms with dy/dx on one side of the equation and solve for dy/dx by dividing by its coefficient.

$$\begin{aligned}\frac{d}{dx}(y\sin x) &= \frac{d}{dx}(x^3) + \frac{d}{dx}(\cos y) \qquad \begin{pmatrix}\text{Use the Product Rule}\\ \text{on the left side.}\end{pmatrix}\\ (\sin x)\frac{dy}{dx} + y\cos x &= 3x^2 - (\sin y)\frac{dy}{dx}\\ (\sin x + \sin y)\frac{dy}{dx} &= 3x^2 - y\cos x\\ \frac{dy}{dx} &= \frac{3x^2 - y\cos x}{\sin x + \sin y}\end{aligned}$$

■

In the examples above the derivatives dy/dx calculated by implicit differentiation depend on y, or on both y and x, rather than just on x. This is to be expected because an equation in x and y can define more than one function of x, and the implicitly calculated derivative must apply to each of the solutions. For example, in Example 2, the derivative $dy/dx = -x/y$ also gives the slope $-3/4$ at the point $(3, 4)$ on the circle. When you use implicit differentiation to find the slope of a curve at a point, you will usually have to know both coordinates of the point.

There are subtle dangers involved in calculating derivatives implicitly. When you use the Chain Rule to differentiate an equation involving y with respect to x, you are automatically assuming that the equation defines y as a differentiable function of x. This need not be the case. To see what can happen, consider the problem of finding $y' = dy/dx$ from the equation

$$x^2 + y^2 = K, \qquad (*)$$

where K is a constant. Just as in Example 2 (where $K = 25$), implicit differentiation gives

$$2x + 2yy' = 0 \qquad \text{or} \qquad y' = -\frac{x}{y}.$$

This formula will give the slope of the curve $(*)$ at any point on the curve where $y \neq 0$. For $K > 0$, $(*)$ represents a circle centred at the origin having radius $\sqrt{K}$. This circle has a finite slope, except at the two points where it crosses the x-axis (where $y = 0$). If $K = 0$, the equation represents only a single point, the origin. The concept of slope of a point is meaningless. For $K < 0$, there are no real points whose coordinates satisfy equation $(*)$, so y' is meaningless here too. The point of this is that being able to calculate y' from a given equation by implicit differentiation does not guarantee that y' actually represents the slope of anything.

If (x_0, y_0) is a point on the graph of the equation $F(x, y) = 0$, there is a theorem that can justify our use of implicit differentiation to find the slope of the graph there. We cannot give a careful statement or proof of this **implicit function theorem** yet (see Section 12.8), but roughly speaking, it says that part of the graph of $F(x, y) = 0$ near (x_0, y_0) is the graph of a function of x that is differentiable at x_0, provided that $F(x, y)$ is a "smooth" function, and that the derivative

$$\frac{d}{dy}F(x_0, y)\bigg|_{y=y_0} \neq 0.$$

For the circle $x^2 + y^2 - K = 0$ (where $K > 0$) this condition says that $2y_0 \neq 0$, which is the condition that the derivative $y' = -x/y$ should exist at (x_0, y_0).

A useful strategy

When you use implicit differentiation to find the value of a derivative at a particular point, it is best to substitute the coordinates of the point immediately after you carry out the differentiation and before you solve for the derivative dy/dx. It is easier to solve an equation involving numbers than one with algebraic expressions.

Example 4 Find an equation of the tangent to $x^2 + xy + 2y^3 = 4$ at $(-2, 1)$.

Solution To find the slope of the tangent we differentiate the given equation implicitly with respect to x. Use the Product Rule to differentiate the xy term:

$$2x + y + xy' + 6y^2y' = 0.$$

Substitute the coordinates $x = -2$, $y = 1$, and solve the resulting equation for y':

$$-4 + 1 - 2y' + 6y' = 0 \qquad \Rightarrow \qquad y' = \frac{3}{4}.$$

The slope of the tangent at $(-2, 1)$ is $3/4$, and its equation is

$$y = \frac{3}{4}(x + 2) + 1 \qquad \text{or} \quad 3x - 4y = -10.$$

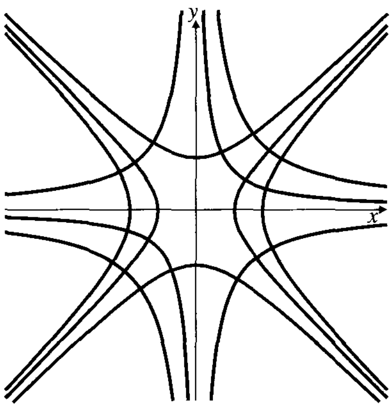

Figure 2.35 Some hyperbolas in the family $x^2 - y^2 = a$ (colour) intersecting some hyperbolas in the family $xy = b$ (black) at right angles

Example 5 Show that for any constants a and b, the curves $x^2 - y^2 = a$ and $xy = b$ intersect at right angles, that is, at any point where they intersect their tangents are perpendicular.

Solution The slope at any point on $x^2 - y^2 = a$ is given by $2x - 2yy' = 0$, or $y' = x/y$. The slope at any point on $xy = b$ is given by $y + xy' = 0$, or $y' = -y/x$. If the two curves (they are both hyperbolas if $a \neq 0$ and $b \neq 0$) intersect at (x_0, y_0), then their slopes at that point are x_0/y_0 and $-y_0/x_0$, respectively. Clearly, these slopes are negative reciprocals, so the tangent line to one curve is the normal line to the other at that point. Hence, the curves intersect at right angles. (See Figure 2.35.)

Higher-Order Derivatives

Example 6 Find $y'' = \dfrac{d^2y}{dx^2}$ if $xy + y^2 = 2x$.

Solution Twice differentiate both sides of the given equation with respect to x:

$$y + xy' + 2yy' = 2$$
$$y' + y' + xy'' + 2(y')^2 + 2yy'' = 0.$$

Now solve these equations for y' and y''.

$$y' = \frac{2 - y}{x + 2y}$$

$$\begin{aligned} y'' &= -\frac{2y' + 2(y')^2}{x + 2y} = -2\,\frac{2 - y}{x + 2y}\,\frac{1 + \dfrac{2 - y}{x + 2y}}{x + 2y} \\ &= -2\,\frac{(2 - y)(x + y + 2)}{(x + 2y)^3} \\ &= -2\,\frac{2x - xy + 2y - y^2 + 4 - 2y}{(x + 2y)^3} = -\frac{8}{(x + 2y)^3}. \end{aligned}$$

We used the given equation to simplify the numerator in the last line.

Remark We can use Maple to calculate derivatives implicitly provided we show explicitly which variable depends on which. For example, we can calculate the value of y'' for the curve $xy + y^3 = 3$ at the point $(2, 1)$ as follows. First we differentiate the equation with respect to x, writing $y(x)$ for y to indicate to Maple that it depends on x.

Note that Maple uses the symbol ∂ instead of d when expressing the derivative in Leibniz form. This is because the expression it is differentiating can involve more than one variable; $(\partial/\partial x)y$ denotes the derivative of y with respect to the specific variable x rather than any other variables on which y may depend. It is called a **partial derivative**. We will study partial derivatives in Chapter 12. For the time being, just regard ∂ as a d.

```
> deq := diff(x*y(x)+(y(x))^3=3, x);
```

$$deq := y(x) + x\left(\frac{\partial}{\partial x}y(x)\right) + 3y(x)^2\left(\frac{\partial}{\partial x}y(x)\right) = 0$$

Now we solve the resulting equation for y':

```
> yp := solve(deq, diff(y(x),x));
```

$$yp := -\frac{y(x)}{x + 3y(x)^2}$$

We can now differentiate *yp* with respect to x to get y''.

```
> ypp := diff(yp,x);
```

$$ypp := -\frac{\frac{\partial}{\partial x}y(x)}{x + 3y(x)^2} + \frac{y(x)\left(1 + 6y(x)\left(\frac{\partial}{\partial x}y(x)\right)\right)}{(x + 3y(x)^2)^2}$$

To get an expression depending only on x and y, we need to substitute the expression obtained for the first derivative into this result. Since the result of this substitution will involve compound fractions, let us simplify the result as well.

```
> ypp := simplify(subs(diff(y(x),x)=yp, ypp);
```

$$ypp := 2\frac{x\,y(x)}{(x + 3y(x)^2)^3}$$

This is y'' expressed as a function of x and y. Now we want to substitute the coordinates $x = 2$, $y(x) = 1$ to get the value of y'' at $(2, 1)$. However, the order of the substitutions is important. *First* we must replace $y(x)$ with 1 and *then* replace x with 2. (If we replace x first, we would have to then replace $y(2)$ rather than $y(x)$ with 1.) Maple's `subs` command makes the substitutions in the order they are written.

```
> subs(y(x)=1, x=2, ypp);
```

$$\frac{4}{125}$$

The General Power Rule

Until now, we have only proven the General Power Rule

$$\frac{d}{dx}x^r = r\,x^{r-1}$$

for integer exponents r and a few special rational exponents such as $r = 1/2$. Using implicit differentiation, we can give the proof for any rational exponent $r = m/n$, where m and n are integers, and $n \neq 0$.

If $y = x^{m/n}$, then $y^n = x^m$. Differentiating implicitly with respect to x, we obtain

$$n\,y^{n-1}\frac{dy}{dx} = m\,x^{m-1}, \qquad \text{so}$$

$$\frac{dy}{dx} = \frac{m}{n}\,x^{m-1}\,y^{1-n} = \frac{m}{n}\,x^{m-1}\,x^{(m/n)(1-n)} = \frac{m}{n}\,x^{m-1+(m/n)-m} = \frac{m}{n}\,x^{(m/n)-1}.$$

Exercises 2.9

In Exercises 1–8, find dy/dx in terms of x and y.

1. $xy - x + 2y = 1$

2. $x^3 + y^3 = 1$

3. $x^2 + xy = y^3$

4. $x^3y + xy^5 = 2$

5. $x^2y^3 = 2x - y$

6. $x^2 + 4(y-1)^2 = 4$

7. $\dfrac{x-y}{x+y} = \dfrac{x^2}{y} + 1$

8. $x\sqrt{x+y} = 8 - xy$

In Exercises 9–16, find an equation of the tangent to the given curve at the given point.

9. $2x^2 + 3y^2 = 5$ at $(1, 1)$

10. $x^2y^3 - x^3y^2 = 12$ at $(-1, 2)$

11. $\dfrac{x}{y} + \left(\dfrac{y}{x}\right)^3 = 2$ at $(-1, -1)$

12. $x + 2y + 1 = \dfrac{y^2}{x-1}$ at $(2, -1)$

13. $2x + y - \sqrt{2}\sin(xy) = \pi/2$ at $\left(\dfrac{\pi}{4}, 1\right)$

14. $\tan(xy^2) = \dfrac{2xy}{\pi}$ at $\left(-\pi, \dfrac{1}{2}\right)$

15. $x\sin(xy - y^2) = x^2 - 1$ at $(1, 1)$

16. $\cos\left(\dfrac{\pi y}{x}\right) = \dfrac{x^2}{y} - \dfrac{17}{2}$ at $(3, 1)$

In Exercises 17–20, find y'' in terms of x and y.

17. $xy = x + y$

18. $x^2 + 4y^2 = 4$

* **19.** $x^3 - y^2 + y^3 = x$

* **20.** $x^3 - 3xy + y^3 = 1$

21. For $x^2 + y^2 = a^2$ show that $y'' = -\dfrac{a^2}{y^3}$.

22. For $Ax^2 + By^2 = C$ show that $y'' = -\dfrac{AC}{B^2y^3}$.

Use Maple or another computer algebra program to find the values requested in Exercises 23–26.

23. Find the slope of $x + y^2 + y\sin x = y^3 + \pi$ at $(\pi, 1)$.

24. Find the slope of $\dfrac{x + \sqrt{y}}{y + \sqrt{x}} = \dfrac{3y - 9x}{x + y}$ at the point $(1, 4)$.

25. If $x + y^5 + 1 = y + x^4 + xy^2$, find d^2y/dx^2 at $(1, 1)$.

26. If $x^3y + xy^3 = 11$, find d^3y/dx^3 at $(1, 2)$.

* **27.** Show that the ellipse $x^2 + 2y^2 = 2$ and the hyperbola $2x^2 - 2y^2 = 1$ intersect at right angles.

* **28.** Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ intersect at right angles if $A^2 \le a^2$ and $a^2 - b^2 = A^2 + B^2$. (This says that the ellipse and the hyperbola have the same foci.)

* **29.** If $z = \tan\dfrac{x}{2}$, show that

$$\frac{dx}{dz} = \frac{2}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}, \quad \text{and } \cos x = \frac{1-z^2}{1+z^2}.$$

* **30.** Use implicit differentiation to find y' if $(x-y)/(x+y) = x/y + 1$. Now show that there are, in fact, no points on that curve, so the derivative you calculated is meaningless. This is another example that demonstrates the dangers of calculating something when you don't know whether or not it exists.

2.10 Antiderivatives and Initial-Value Problems

Throughout this chapter we have been concerned with the problem of finding the derivative f' of a given function f. The reverse problem—given the derivative f', find f—is also interesting and important. It is the problem studied in *integral calculus* and is generally harder to solve than the problem of finding a derivative. We will take a preliminary look at this problem in this section and will return to it in more detail in Chapter 5.

Antiderivatives

We begin by defining an antiderivative of a function f to be a function F whose derivative is f. It is appropriate to require that $F'(x) = f(x)$ on an *interval*.

DEFINITION 7

An **antiderivative** of a function f on an interval I is another function F satisfying

$$F'(x) = f(x) \quad \text{for } x \text{ in } I.$$

Example 1

(a) $F(x) = x$ is an antiderivative of the function $f(x) = 1$ on any interval because $F'(x) = 1 = f(x)$ everywhere.

(b) $G(x) = \frac{1}{2}x^2$ is an antiderivative of the function $g(x) = x$ on any interval because $G'(x) = \frac{1}{2}(2x) = x = g(x)$ everywhere.

(c) $R(x) = -\frac{1}{3}\cos(3x)$ is an antiderivative of $r(x) = \sin(3x)$ on any interval because $R'(x) = -\frac{1}{3}(-3\sin(3x)) = \sin(3x) = r(x)$ everywhere.

(d) $F(x) = -1/x$ is an antiderivative of $f(x) = 1/x^2$ on any interval not containing $x = 0$ because $F'(x) = 1/x^2 = f(x)$ everywhere except at $x = 0$.

Antiderivatives are not unique; indeed, if C is any constant, then $F(x) = x + C$ is an antiderivative of $f(x) = 1$ on any interval. You can always add a constant to an antiderivative F of a function f on an interval and get another antiderivative of f. More importantly, *all* antiderivatives of f on an interval can be obtained by adding constants to any particular one. If F and G are both antiderivatives of f on an interval I, then

$$\frac{d}{dx}\big(G(x) - F(x)\big) = f(x) - f(x) = 0$$

on I, so $G(x) - F(x) = C$ (a constant) on I by Theorem 13 of Section 2.6. Thus $G(x) = F(x) + C$ on I.

Note that neither this conclusion nor Theorem 13 is valid over a set that is not an interval. For example, the derivative of

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is 0 for all $x \neq 0$, but $\operatorname{sgn} x$ is not constant for all $x \neq 0$. $\operatorname{sgn} x$ has *different* constant values on the two intervals $]-\infty, 0[$ and $]0, \infty[$ comprising its domain.

The Indefinite Integral

The *general antiderivative* of a function $f(x)$ on an interval I is $F(x) + C$, where $F(x)$ is any particular antiderivative of $f(x)$ on I and C is a constant. This general antiderivative is called the indefinite integral of $f(x)$ on I and is denoted $\int f(x)\,dx$.

DEFINITION 8

The **indefinite integral** of $f(x)$ on interval I is

$$\int f(x)\,dx = F(x) + C \qquad \text{on } I,$$

provided $F'(x) = f(x)$ for all x in I.

The symbol $\int$ is called an **integral sign**. It is shaped like an elongated "S" for reasons that will only become apparent when we study the *definite integral* in Chapter 5. Just as you regard dy/dx as a single symbol representing the derivative of y with respect to x, so you should regard $\int f(x)\,dx$ as a single symbol representing the indefinite integral (general antiderivative) of f with respect to x. The constant C is called a **constant of integration**.

Example 2

(a) $\displaystyle\int x\,dx = \frac{1}{2}x^2 + C$ on any interval.

(b) $\displaystyle\int (x^3 - 5x^2 + 7)\,dx = \frac{1}{4}x^4 - \frac{5}{3}x^3 + 7x + C$ on any interval.

(c) $\displaystyle\int \left(\frac{1}{x^2} + \frac{2}{\sqrt{x}}\right) dx = -\frac{1}{x} + 4\sqrt{x} + C$ on any interval to the right of $x = 0$.

All three formulas above can be checked by differentiating the right-hand sides. ■

Finding antiderivatives is generally more difficult than finding derivatives; many functions do not have antiderivatives that can be expressed as combinations of finitely many elementary functions. However, *every formula for a derivative can be rephrased as a formula for an antiderivative.* For instance,

$$\frac{d}{dx}\sin x = \cos x; \qquad \text{therefore,} \quad \int \cos x\,dx = \sin x + C.$$

We will develop several techniques for finding antiderivatives in later chapters. Until then, we must content ourselves with being able to write a few simple antiderivatives based on the known derivatives of elementary functions:

(a) $\displaystyle\int dx = \int 1\,dx = x + C$ (b) $\displaystyle\int x\,dx = \frac{x^2}{2} + C$

(c) $\displaystyle\int x^2\,dx = \frac{x^3}{3} + C$ (d) $\displaystyle\int \frac{1}{x^2}\,dx = \int \frac{dx}{x^2} = -\frac{1}{x} + C$

(e) $\displaystyle\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$ (f) $\displaystyle\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$

(g) $\displaystyle\int \sin x\,dx = -\cos x + C$ (h) $\displaystyle\int \cos x\,dx = \sin x + C$

(i) $\displaystyle\int \sec^2 x\,dx = \tan x + C$ (j) $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

(k) $\displaystyle\int \sec x \tan x\,dx = \sec x + C$ (l) $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

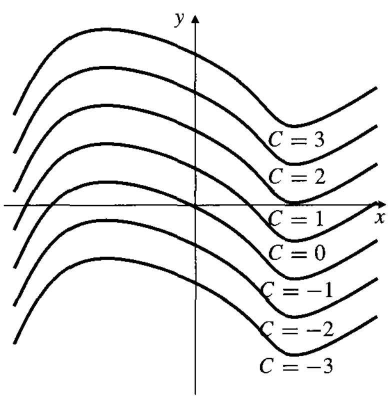

Figure 2.36 Graphs of various antiderivatives of the same function

Observe that formulas (a)–(e) are special cases of formula (f). For the moment, r must be rational in (f), but this restriction will be removed later.

The rule for differentiating sums and constant multiples of functions translates into a similar rule for antiderivatives, as reflected in parts (b) and (c) of Example 2 above.

The graphs of the different antiderivatives of the same function on the same interval are vertically displaced versions of the same curve, as shown in Figure 2.36.

In general, only one of these curves will pass through any given point, so we can obtain a unique antiderivative of a given function on an interval by requiring the antiderivative to take a prescribed value at a particular point x.

Example 3 Find the function $f(x)$ whose derivative is $f'(x) = 6x^2 - 1$ for all real x and for which $f(2) = 10$.

Solution Since $f'(x) = 6x^2 - 1$, we have

$$f(x) = \int (6x^2 - 1)\, dx = 2x^3 - x + C$$

for some constant C. Since $f(2) = 10$, we have

$$10 = f(2) = 16 - 2 + C.$$

Thus $C = -4$ and $f(x) = 2x^3 - x - 4$. (By direct calculation we can verify that $f'(x) = 6x^2 - 1$ and $f(2) = 10$.) ■

Example 4 Find the function $g(t)$ whose derivative is $\dfrac{t+5}{t^{3/2}}$ and whose graph passes through the point $(4, 1)$.

Solution We have

$$\begin{aligned} g(t) &= \int \frac{t+5}{t^{3/2}}\, dt \\ &= \int (t^{-1/2} + 5t^{-3/2})\, dt \\ &= 2t^{1/2} - 10t^{-1/2} + C \end{aligned}$$

Since the graph of $y = g(t)$ must pass through $(4, 1)$, we require that

$$1 = g(4) = 4 - 5 + C.$$

Hence, $C = 2$ and

$$g(t) = 2t^{1/2} - 10t^{-1/2} + 2 \qquad \text{for } t > 0.$$

■

Differential Equations and Initial-Value Problems

A **differential equation** (abbreviated DE) is an equation involving one or more derivatives of an unknown function. Any function whose derivatives satisfy the differential equation *identically on an interval* is called a **solution** of the equation on that interval. For instance, the function $y = x^3 - x$ is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 - 1$$

on the whole real line. This differential equation has more than one solution; in fact, $y = x^3 - x + C$ is a solution for any value of the constant C.

Example 5 Show that for any constants A and B, the function $y = Ax^3 + B/x$ is a solution of the differential equation $x^2y'' - xy' - 3y = 0$ on any interval not containing 0.

Solution If $y = Ax^3 + B/x$, then for $x \neq 0$ we have

$$y' = 3Ax^2 - B/x^2 \quad \text{and} \quad y'' = 6Ax + 2B/x^3.$$

Therefore,

$$x^2y'' - xy' - 3y = 6Ax^3 + \frac{2B}{x} - 3Ax^3 + \frac{B}{x} - 3Ax^3 - \frac{3B}{x} = 0,$$

provided $x \neq 0$. This is what had to be proved. ■

The **order** of a differential equation is the order of the highest-order derivative appearing in the equation. The DE in Example 5 is a *second-order* DE since it involves y'' and no higher derivatives of y. Note that the solution verified in Example 5 involves two arbitrary constants, A and B. This solution is called a **general solution** to the equation since it can be shown that every solution is of this form for some choice of the constants A and B. A **particular solution** of the equation is obtained by assigning specific values to these constants. The general solution of an nth-order differential equation typically involves n arbitrary constants.

An **initial-value problem** (abbreviated IVP) is a problem that consists of:

(i) a differential equation (to be solved for an unknown function) and

(ii) prescribed values for the solution and enough of its derivatives at a particular point (the initial point) to determine values for all the arbitrary constants in the general solution of the DE and so yield a particular solution.

Remark It is common to use the same symbol, say y, to denote both the dependent variable and the function that is the solution to a DE or an IVP; that is, we call the solution function $y = y(x)$ rather than $y = f(x)$.

Remark The solution of an IVP is valid in the largest interval containing the initial point where the solution function is defined.

Example 6 Use the result of Example 5 to solve the following initial-value problem.

$$\begin{cases} x^2y'' - xy' - 3y = 0 & (x > 0) \\ y(1) = 2 \\ y'(1) = -6 \end{cases}$$

Solution As shown in Example 5, the DE $x^2y'' - xy' - 3y = 0$ has solution $y = Ax^3 + B/x$, which has derivative $y' = 3Ax^2 - B/x^2$. At $x = 1$ we must have $y = 2$ and $y' = -6$. Therefore,

$$\begin{aligned} A+B &= 2 \\ 3A-B &= -6. \end{aligned}$$

Solving these two linear equations for A and B, we get $A = -1$ and $B = 3$. Hence, $y = -x^3 + 3/x$ for $x > 0$ is the solution of the IVP. ∎

One of the simplest kinds of differential equation is the equation

$$\frac{dy}{dx} = f(x),$$

which is to be solved for y as a function of x. Evidently the solution is

$$y = \int f(x)\,dx.$$

Our ability to find the unknown function $y(x)$ depends on our ability to find an antiderivative of f.

Example 7 Solve the initial-value problem

$$\begin{cases} y' = \dfrac{3+2x^2}{x^2} \\ y(-2) = 1. \end{cases}$$

Where is the solution valid?

Solution

$$y = \int \left(\frac{3}{x^2} + 2\right) dx = -\frac{3}{x} + 2x + C$$

$$1 = y(-2) = \frac{3}{2} - 4 + C$$

Therefore, $C = \frac{7}{2}$ and

$$y = -\frac{3}{x} + 2x + \frac{7}{2}.$$

Although the solution function appears to be defined for all x except 0, it is only a solution of the given initial-value problem for $x < 0$. This is because $]-\infty, 0[$ is the largest interval that contains the initial point -2 but not the point $x = 0$, where the solution y is undefined. ∎

Example 8 Solve the second-order IVP

$$\begin{cases} y'' = \sin x \\ y(\pi) = 2 \\ y'(\pi) = -1. \end{cases}$$

Solution Since $(y')' = y'' = \sin x$, we have

$$y'(x) = \int \sin x \, dx = -\cos x + C_1.$$

The initial condition for y' gives

$$-1 = y'(\pi) = -\cos\pi + C_1 = 1 + C_1,$$

so that $C_1 = -2$ and $y'(x) = -(\cos x + 2)$. Thus

$$\begin{aligned} y(x) &= -\int (\cos x + 2)\, dx \\ &= -\sin x - 2x + C_2. \end{aligned}$$

The initial condition for y now gives

$$2 = y(\pi) = -\sin\pi - 2\pi + C_2 = -2\pi + C_2,$$

so that $C_2 = 2 + 2\pi$. The solution to the given IVP is

$$y = 2 + 2\pi - \sin x - 2x$$

and is valid for all x. ■

Differential equations and initial-value problems are of great importance in applications of calculus, especially for expressing in mathematical form certain laws of nature that involve rates of change of quantities. A large portion of the total mathematical endeavour of the last two hundred years has been devoted to their study. They are usually treated in separate courses on differential equations, but we will discuss them from time to time in this book when appropriate. Throughout this book, exercises about differential equations and initial-value problems are designated with the symbol ◈.

Exercises 2.10

In Exercises 1–14, find the given indefinite integrals.

1. $\int 5\, dx$
2. $\int x^2\, dx$
3. $\int \sqrt{x}\, dx$
4. $\int x^{12}\, dx$
5. $\int x^3\, dx$
6. $\int (x + \cos x)\, dx$
7. $\int \tan x \cos x\, dx$
8. $\int \frac{1 + \cos^3 x}{\cos^2 x}\, dx$
9. $\int (a^2 - x^2)\, dx$
10. $\int (A + Bx + Cx^2)\, dx$
11. $\int (2x^{1/2} + 3x^{1/3})\, dx$
12. $\int \frac{6(x-1)}{x^{4/3}}\, dx$
13. $\int \left(\frac{x^3}{3} - \frac{x^2}{2} + x - 1\right) dx$
14. $105 \int (1 + t^2 + t^4 + t^6)\, dt$

In Exercises 15–22, find the given indefinite integrals. This may require guessing the form of an antiderivative and then checking by differentiation. For instance, you might suspect that

$\int \cos(5x-2)\,dx = k\sin(5x-2)+C$ for some k. Differentiating the answer shows that k must be $1/5$.

15. $\displaystyle\int \cos(2x)\,dx$

16. $\displaystyle\int \sin\left(\frac{x}{2}\right)\,dx$

∗ 17. $\displaystyle\int \frac{dx}{(1+x)^2}$

∗ 18. $\displaystyle\int \sec(1-x)\tan(1-x)\,dx$

∗ 19. $\displaystyle\int \sqrt{2x+3}\,dx$

∗ 20. $\displaystyle\int \frac{4}{\sqrt{x+1}}\,dx$

21. $\displaystyle\int 2x\sin(x^2)\,dx$

∗ 22. $\displaystyle\int \frac{2x}{\sqrt{x^2+1}}\,dx$

Use trigonometric identities such as $\sec^2 x = 1+\tan^2 x$, $\sin(2x) = 2\sin x\cos x$, and $\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x$ to help you evaluate the indefinite integrals in Exercises 23–26.

∗ 23. $\displaystyle\int \tan^2 x\,dx$

∗ 24. $\displaystyle\int \sin x\cos x\,dx$

∗ 25. $\displaystyle\int \cos^2 x\,dx$

∗ 26. $\displaystyle\int \sin^2 x\,dx$

Differential equations

In Exercises 27–42, find the solution $y = y(x)$ to the given initial-value problem. On what interval is the solution valid? (Note that exercises involving differential equations are prefixed with the symbol ◈.)

◈ 27. $\begin{cases} y' = x-2 \\ y(0) = 3 \end{cases}$

◈ 28. $\begin{cases} y' = x^{-2} - x^{-3} \\ y(-1) = 0 \end{cases}$

◈ 29. $\begin{cases} y' = 3\sqrt{x} \\ y(4) = 1 \end{cases}$

◈ 30. $\begin{cases} y' = x^{1/3} \\ y(0) = 5 \end{cases}$

◈ 31. $\begin{cases} y' = Ax^2 + Bx + C \\ y(1) = 1 \end{cases}$

◈ 32. $\begin{cases} y' = x^{-9/7} \\ y(1) = -4 \end{cases}$

◈ 33. $\begin{cases} y' = \cos x \\ y(\pi/6) = 2 \end{cases}$

◈ 34. $\begin{cases} y' = \sin(2x) \\ y(\pi/2) = 1 \end{cases}$

◈ 35. $\begin{cases} y' = \sec^2 x \\ y(0) = 1 \end{cases}$

◈ 36. $\begin{cases} y' = \sec^2 x \\ y(\pi) = 1 \end{cases}$

◈ 37. $\begin{cases} y'' = 2 \\ y'(0) = 5 \\ y(0) = -3 \end{cases}$

◈ 38. $\begin{cases} y'' = x^{-4} \\ y'(1) = 2 \\ y(1) = 1 \end{cases}$

◈ 39. $\begin{cases} y'' = x^3 - 1 \\ y'(0) = 0 \\ y(0) = 8 \end{cases}$

◈ 40. $\begin{cases} y'' = 5x^2 - 3x^{-1/2} \\ y'(1) = 2 \\ y(1) = 0 \end{cases}$

◈ 41. $\begin{cases} y'' = \cos x \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

◈ 42. $\begin{cases} y'' = x + \sin x \\ y(0) = 2 \\ y'(0) = 0 \end{cases}$

◈ 43. Show that for any constants A and B the function $y = y(x) = Ax + B/x$ satisfies the *second-order differential equation* $x^2y'' + xy' - y = 0$ for $x \neq 0$. Find a function y satisfying the initial-value problem:

$$\begin{cases} x^2y'' + xy' - y = 0 \qquad (x > 0) \\ y(1) = 2 \\ y'(1) = 4. \end{cases}$$

◈ 44. Show that for any constants A and B the function $y = Ax^{r_1} + Bx^{r_2}$ satisfies, for $x > 0$, the differential equation $ax^2y'' + bxy' + cy = 0$, provided that r_1 and r_2 are two distinct rational roots of the quadratic equation $ar(r-1) + br + c = 0$.

Use the result of Exercise 44 to solve the initial-value problems in Exercises 45–46 on the interval $x > 0$.

◈ 45. $\begin{cases} 4x^2y'' + 4xy' - y = 0 \\ y(4) = 2 \\ y'(4) = -2 \end{cases}$

◈ 46. $\begin{cases} x^2y'' - 6y = 0 \\ y(1) = 1 \\ y'(1) = 1 \end{cases}$

2.11 Velocity and Acceleration

Velocity and Speed

Suppose that an object is moving along a straight line (say the x-axis) so that its position x is a function of time t, say $x = x(t)$. (We are using x to represent both the dependent variable and the function.) Suppose we are measuring x in metres and t in seconds. The **average velocity** of the object over the time interval $[t, t+h]$ is the change in position divided by the change in time, that is, the Newton quotient

$$v_{\text{average}} = \frac{\Delta x}{\Delta t} = \frac{x(t+h) - x(t)}{h} \text{ m/s.}$$

The **velocity** $v(t)$ of the object at time t is the limit of this average velocity as $h \to 0$. Thus it is the rate of change (the derivative) of position with respect to time:

$$\text{Velocity:} \quad v(t) = \frac{dx}{dt} = x'(t).$$

Besides telling us how fast the object is moving, the velocity also tells us in which direction it is moving. If $v(t) > 0$, then x is increasing, so the object is moving to the right; if $v(t) < 0$, then x is decreasing, so the object is moving to the left. At a critical point of x, that is, a time t when $v(t) = 0$, the object is instantaneously at rest—at that instant it is not moving in either direction.

We distinguish between the term *velocity* (which involves direction of motion as well as the rate) and **speed**, which only involves the rate, and not the direction. The speed is the absolute value of the velocity:

$$\text{Speed:} \quad s(t) = |v(t)| = \left|\frac{dx}{dt}\right|.$$

A speedometer gives us the speed an automobile is moving; it does not give the velocity. The speedometer does not start to show negative values if the automobile turns around and heads in the opposite direction.

Example 1

(a) Determine the velocity $v(t)$ at time t of an object moving along the x-axis so that at time t its position is given by

$$x = v_0 t + \frac{1}{2}at^2,$$

where v_0 and a are constants.

(b) Draw the graph of the function $v(t)$, and show that the area under the graph and above the t-axis, over the interval $[t_1, t_2]$, is equal to the distance the object travels in that time interval.

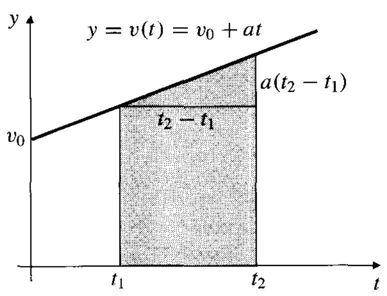

Figure 2.37 The shaded area equals the distance travelled between t_1 and t_2

Solution The velocity is given by

$$v(t) = \frac{dx}{dt} = v_0 + at.$$

Its graph is a straight line with slope a and intercept v_0 on the vertical (velocity) axis. The area under the graph (shaded in Figure 2.37) is the sum of the areas of a rectangle and a triangle. Each has base $t_2 - t_1$. The rectangle has height $v(t_1) = v_0 + at_1$, and the triangle has height $a(t_2 - t_1)$. (Why?) Thus the shaded area is equal to

$$\begin{aligned}\text{Area} &= (t_2 - t_1)(v_0 + at_1) + \frac{1}{2}(t_2 - t_1)[a(t_2 - t_1)] \\ &= (t_2 - t_1)\left[v_0 + at_1 + \frac{a}{2}(t_2 - t_1)\right] \\ &= (t_2 - t_1)\left[v_0 + \frac{a}{2}(t_2 + t_1)\right] \\ &= v_0(t_2 - t_1) + \frac{a}{2}(t_2^2 - t_1^2) \\ &= x(t_2) - x(t_1),\end{aligned}$$

which is the distance travelled by the object between times t_1 and t_2. ■

Remark In Example 1 we differentiated the position x to get the velocity v and then used the area under the velocity graph to recover information about the position. It appears that there is a connection between finding areas and finding functions that have given derivatives (i.e., finding antiderivatives). This connection, which we will explore in Chapter 5, is perhaps the most important idea in calculus!

Acceleration

The derivative of the velocity also has a useful interpretation. The rate of change of the velocity with respect to time is the **acceleration** of the moving object. It is measured in units of distance/time^2. The value of the acceleration at time t is

$$\text{Acceleration:} \quad a(t) = v'(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

The acceleration is the *second derivative* of the position. If $a(t) > 0$, the velocity is increasing. This does not necessarily mean that the speed is increasing; if the object is moving to the left ($v(t) < 0$) and accelerating to the right ($a(t) > 0$), then it is actually slowing down. The object is speeding up only when the velocity and acceleration have the same sign.

Table 2. Velocity, acceleration, and speed

If velocity is	and acceleration is	then object is	and speed is
positive	positive	moving right	increasing
positive	negative	moving right	decreasing
negative	positive	moving left	decreasing
negative	negative	moving left	increasing

If $a(t_0) = 0$, then the velocity and the speed are stationary at t_0. If $a(t) = 0$ during an interval of time, then the velocity is unchanging and, therefore, constant over that interval.

Example 2 A point P moves along the x-axis in such a way that its position at time t s is given by

$$x = 2t^3 - 15t^2 + 24t \text{ ft.}$$

(a) Find the velocity and acceleration of P at time t.

(b) In which direction and how fast is P moving at $t = 2$ s? Is it speeding up or slowing down at that time?

(c) When is P instantaneously at rest? When is its speed instantaneously not changing?

(d) When is P moving to the left? to the right?

(e) When is P speeding up? slowing down?

Solution

(a) The velocity and acceleration of P at time t are

$$v = \frac{dx}{dt} = 6t^2 - 30t + 24 = 6(t-1)(t-4) \text{ ft/s} \quad \text{and}$$
$$a = \frac{dv}{dt} = 12t - 30 = 6(2t-5) \text{ ft/s}^2.$$

(b) At $t = 2$ we have $v = -12$ and $a = -6$. Thus P is moving to the left with speed 12 ft/s, and, since the velocity and acceleration are both negative, its speed is increasing.

(c) P is at rest when $v = 0$, that is, when $t = 1$ s or $t = 4$ s. Its speed is unchanging when $a = 0$, that is, at $t = 5/2$ s.

(d) The velocity is continuous for all t so, by the Intermediate-Value Theorem, has a constant sign on the intervals between the points where it is 0. By examining the values of $v(t)$ at $t = 0$, 2, and 5 (or by analyzing the signs of the factors $(t-1)$ and $(t-4)$ in the expression for $v(t)$), we conclude that $v(t) < 0$ (and P is moving to the left) on time interval $(1, 4)$. $v(t) > 0$ (and P is moving to the right) on time intervals $]-\infty, 1[$ and $]4, \infty[$.

(e) The acceleration a is negative for $t < 5/2$ and is positive for $t > 5/2$. Table 3 combines this information with information about v to show where P is speeding up and slowing down.

Table 3. Data for Example 2

Interval	$v(t)$ is	$a(t)$ is	P is
$]-\infty, 1[$	positive	negative	slowing down
$]1, 5/2[$	negative	negative	speeding up
$]5/2, 4[$	negative	positive	slowing down
$]4, \infty[$	positive	positive	speeding up

The motion of P is shown in Figure 2.38. ∎

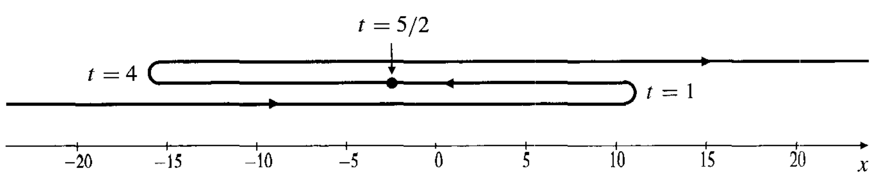

Figure 2.38 The motion of P in Example 2

Example 3 An object is hurled upward from the roof of a building 10 m high. It rises and then falls back; its height above ground t s after it is thrown is

$$y = -4.9t^2 + 8t + 10 \text{ m},$$

until it strikes the ground. What is the greatest height above the ground that the object attains? With what speed does the object strike the ground?

Solution Refer to Figure 2.39. The vertical velocity at time t during flight is

$$v(t) = -2(4.9)t + 8 = -9.8t + 8 \text{ m/s}.$$

The object is rising when $v > 0$, that is, when $0 < t < 8/9.8$, and is falling for $t > 8/9.8$. Thus the object is at its maximum height at time $t = 8/9.8 \approx 0.8163$ s, and this maximum height is

$$y_{\max} = -4.9\left(\frac{8}{9.8}\right)^2 + 8\left(\frac{8}{9.8}\right) + 10 \approx 13.27 \text{ m}.$$

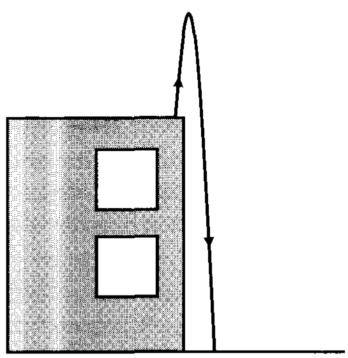

Figure 2.39

The time t at which the object strikes the ground is the positive root of the quadratic equation obtained by setting $y = 0$,

$$-4.9t^2 + 8t + 10 = 0,$$

namely,

$$t = \frac{-8 - \sqrt{64 + 196}}{-9.8} \approx 2.462 \text{ s}.$$

The velocity at this time is $v = -(9.8)(2.462) + 8 \approx -16.12$. Thus the object strikes the ground with a speed of about 16.12 m/s. ■

Falling Under Gravity

According to Newton's Second Law of Motion, a rock of mass m acted on by an unbalanced force F will experience an acceleration a proportional to and in the same direction as F; with appropriate units of force, $F = ma$. If the rock is sitting on the ground, it is acted on by two forces: the force of gravity acting downward and reaction of the ground acting upward. These forces balance, so there is no resulting acceleration. On the other hand, if the rock is up in the air and is unsupported, the gravitational force on it will be unbalanced and the rock will experience downward acceleration. It will fall.

According to Newton's Universal Law of Gravitation, the force by which the earth attracts the rock is proportional to the mass m of the rock and inversely proportional to the square of its distance r from the centre of the earth: $F = km/r^2$. If the relative change $\Delta r/r$ is small, as will be the case if the rock remains near the surface of the earth, then $F = mg$, where $g = k/r^2$ is approximately constant. It follows that $ma = F = mg$, and the rock experiences *constant* downward acceleration g. Since g does not depend on m, all objects experience the same acceleration when falling near the surface of the earth, provided we ignore air resistance and any other forces that may be acting on them. Newton's laws therefore imply that if the height of such an object at time t is $y(t)$, then

$$\frac{d^2y}{dt^2} = -g.$$

The negative sign is needed because the gravitational acceleration is downward, the opposite direction to that of increasing y. Physical experiments give the following approximate values for g at the surface of the earth:

$$g = 32 \text{ ft/s}^2 \quad \text{or} \quad g = 9.8 \text{ m/s}^2.$$

Example 4 A rock falling freely near the surface of the earth is subject to a constant downward acceleration g, if the effect of air resistance is neglected. If the height and velocity of the rock are y_0 and v_0 at time $t = 0$, find the height $y(t)$ of the rock at any later time t until the rock strikes the ground.

Solution This example asks for a solution $y(t)$ to the second-order initial-value problem:

$$\begin{cases} y''(t) = -g \\ y(0) = y_0 \\ y'(0) = v_0. \end{cases}$$

We have

$$y'(t) = -\int g\,dt = -gt + C_1$$
$$v_0 = y'(0) = 0 + C_1.$$

Thus $C_1 = v_0$.

$$y'(t) = -gt + v_0$$
$$y(t) = \int (-gt + v_0)dt = -\frac{1}{2}gt^2 + v_0t + C_2$$
$$y_0 = y(0) = 0 + 0 + C_2.$$

Thus $C_2 = y_0$. Finally, therefore,

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0.$$

■

Example 5 A ball is thrown down with an initial speed of 20 ft/s from the top of a cliff, and it strikes the ground at the bottom of the cliff after 5 s. How high is the cliff?

Solution We will apply the result of Example 4. Here we have $g = 32$ ft/s^2, $v_0 = -20$ ft/s, and y_0 is the unknown height of the cliff. The height of the ball t s after it is thrown down is

$$y(t) = -16t^2 - 20t + y_0 \text{ ft.}$$

At $t = 5$ the ball reaches the ground, so $y(5) = 0$:

$$0 = -16(25) - 20(5) + y_0 \qquad \Rightarrow \qquad y_0 = 500.$$

The cliff is 500 ft high. ∎

Example 6 **(Stopping distance)** A car is travelling at 72 km/h. At a certain instant its brakes are applied to produce a constant deceleration of 0.8 m/s^2. How far does the car travel before coming to a stop?

Solution Let $s(t)$ be the distance the car travels in the t seconds after the brakes are applied. Then $s''(t) = -0.8$ (m/s^2), so the velocity at time t is given by

$$s'(t) = \int -0.8\, dt = -0.8t + C_1 \quad \text{m/s.}$$

Since $s'(0) = 72$ km/h $= 72 \times 1000/3600 = 20$ m/s, we have $C_1 = 20$. Thus,

$$s'(t) = 20 - 0.8t$$

and

$$s(t) = \int (20 - 0.8t)\, dt = 20t - 0.4t^2 + C_2.$$

Since $s(0) = 0$, we have $C_2 = 0$ and $s(t) = 20t - 0.4t^2$. When the car has stopped, its velocity will be 0. Hence, the stopping time is the solution t of the equation

$$0 = s'(t) = 20 - 0.8t,$$

that is, $t = 25$ s. The distance travelled during deceleration is $s(25) = 250$ m. ∎

Exercises 2.11

In Exercises 1–4, a point moves along the x-axis so that its position x at time t is specified by the given function. In each case determine the following:

(a) the time intervals on which the point is moving to the right and (b) to the left;

(c) the time intervals on which the point is accelerating to the right and (d) to the left;

(e) the time intervals when the particle is speeding up and (f) slowing down;

(g) the acceleration at times when the velocity is zero;

(h) the average velocity over the time interval [0, 4].

1. $x = t^2 - 4t + 3$

2. $x = 4 + 5t - t^2$

3. $x = t^3 - 4t + 1$

4. $x = \dfrac{t}{t^2 + 1}$

5. A ball is thrown upward from ground level with an initial speed of 9.8 m/s so that its height in metres after t s is given by $y = 9.8t - 4.9t^2$. What is the acceleration of the ball at any time t? How high does the ball go? How fast is it moving when it strikes the ground?

6. A ball is thrown downward from the top of a 100-metre-high tower with an initial speed of 2 m/s. Its height in metres above the ground t s later is $y = 100 - 2t - 4.9t^2$. How long does it take to reach the ground? What is its average velocity during the fall? At what instant is its velocity equal to its average velocity?

* **7.** **(Takeoff distance)** The distance an aircraft travels along a runway before takeoff is given by $D = t^2$, where D is measured in metres from the starting point, and t is measured in seconds from the time the brake is released. If the aircraft will become airborne when its speed reaches 200 km/h, how long will it take to become airborne, and what distance will it travel in that time?

8. **(Projectiles on Mars)** A projectile fired upward from the surface of the earth falls back to the ground after 10 s. How long would it take to fall back to the surface if it is fired upward on Mars with the same initial velocity? $g_{\text{Mars}} = 3.72\ \text{m/s}^2$.

9. A ball is thrown upward with initial velocity v_0 m/s and reaches a maximum height of h m. How high would it have gone if its initial velocity was $2v_0$? How fast must it be thrown upward to achieve a maximum height of $2h$ m?

10. How fast would the ball in the previous exercise have to be thrown upward on Mars in order to achieve a maximum height of $3h$ m?

11. A rock falls from the top of a cliff and hits the ground at the base of the cliff at a speed of 160 ft/s. How high is the cliff?

12. A rock is thrown down from the top of a cliff with the initial speed of 32 ft/s and hits the ground at the base of the cliff at a speed of 160 ft/s. How high is the cliff?

13. **(Distance travelled while braking)** With full brakes applied, a freight train can decelerate at a constant rate of $1/6\ \text{m/s}^2$. How far will the train travel while braking to a full stop from an initial speed of 60 km/h?

* **14.** Show that if the position x of a moving point is given by a quadratic function of t, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of that time interval.

* **15.** **(Piecewise motion)** The position of an object moving along the s-axis is given at time t by

$$s = \begin{cases} t^2 & \text{if } 0 \le t \le 2 \\ 4t - 4 & \text{if } 2 < t < 8 \\ -68 + 20t - t^2 & \text{if } 8 \le t \le 10. \end{cases}$$

Determine the velocity and acceleration at any time t. Is the velocity continuous? Is the acceleration continuous? What is the maximum velocity and when is it attained?

(Rocket flight with limited fuel) Figure 2.40 shows the velocity v in ft/s of a small rocket that was fired from the top of a tower at time $t = 0$ (t in seconds), accelerated with constant upward acceleration until its fuel was used up, then fell back to the ground at the foot of the tower. The whole flight lasted 14 s. Exercises 16–19 refer to this rocket.

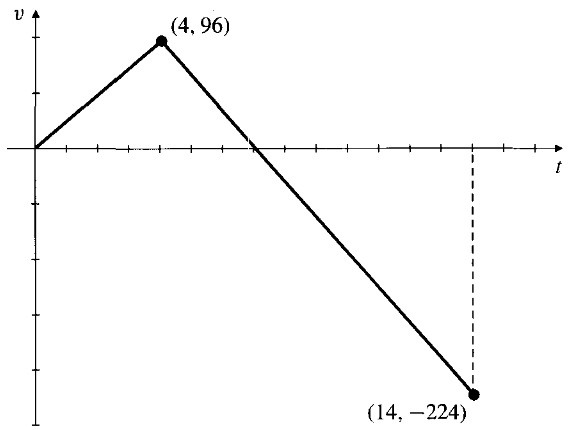

Figure 2.40

16. What was the acceleration of the rocket while its fuel lasted?

17. How long was the rocket rising?

* **18.** What is the maximum height above ground that the rocket reached?

* **19.** How high was the tower from which the rocket was fired?

20. Redo Example 6 using instead a nonconstant deceleration, $s''(t) = -t\ \text{m/s}^2$.

Chapter Review

Key Ideas

- **What do the following statements and phrases mean?**
  - ◇ Line L is tangent to curve C at point P.
  - ◇ the Newton quotient of $f(x)$ at $x = a$
  - ◇ the derivative $f'(x)$ of the function $f(x)$
  - ◇ f is differentiable at $x = a$.
  - ◇ the slope of the graph $y = f(x)$ at $x = a$
  - ◇ f is increasing (or decreasing) on interval I.
  - ◇ f is nondecreasing (or nonincreasing) on interval I.
  - ◇ the average rate of change of $f(x)$ on $[a, b]$
  - ◇ the rate of change of $f(x)$ at $x = a$
  - ◇ c is a critical point of $f(x)$.
  - ◇ the second derivative of $f(x)$ at $x = a$
  - ◇ an antiderivative of f on interval I
  - ◇ the indefinite integral of f on interval I
  - ◇ differential equation
  - ◇ initial-value problem

◇ velocity ◇ speed ◇ acceleration

- **State the following differentiation rules:**

◇ the rule for differentiating a sum of functions

◇ the rule for differentiating a constant multiple of a function

◇ the Product Rule ◇ the Reciprocal Rule

◇ the Quotient Rule ◇ the Chain Rule

- **State the Mean-Value Theorem.**
- **State the Generalized Mean-Value Theorem.**
- **State the derivatives of the following functions:**

◇ x ◇ x^2 ◇ $1/x$ ◇ $\sqrt{x}$

◇ x^n ◇ $|x|$ ◇ $\sin x$ ◇ $\cos x$

◇ $\tan x$ ◇ $\cot x$ ◇ $\sec x$ ◇ $\csc x$

- **What is a proof by mathematical induction?**

Review Exercises

Use the definition of derivative to calculate the derivatives in Exercises 1–4.

1. $\dfrac{dy}{dx}$ if $y = (3x+1)^2$ **2.** $\dfrac{d}{dx}\sqrt{1-x^2}$

3. $f'(2)$ if $f(x) = \dfrac{4}{x^2}$ **4.** $g'(9)$ if $g(t) = \dfrac{t-5}{1+\sqrt{t}}$

5. Find the tangent to $y = \cos(\pi x)$ at $x = 1/6$.

6. Find the normal to $y = \tan(x/4)$ at $x = \pi$.

Calculate the derivatives of the functions in Exercises 7–12.

7. $\dfrac{1}{x - \sin x}$ **8.** $\dfrac{1 + x + x^2 + x^3}{x^4}$

9. $(4 - x^{2/5})^{-5/2}$ **10.** $\sqrt{2 + \cos^2 x}$

11. $\tan\theta - \theta\sec^2\theta$ **12.** $\dfrac{\sqrt{1+t^2}-1}{\sqrt{1+t^2}+1}$

Evaluate the limits in Exercises 13–16 by interpreting each as a derivative.

13. $\displaystyle\lim_{h\to 0}\frac{(x+h)^{20} - x^{20}}{h}$ **14.** $\displaystyle\lim_{x\to 2}\frac{\sqrt{4x+1}-3}{x-2}$

15. $\displaystyle\lim_{x\to\pi/6}\frac{\cos(2x) - (1/2)}{x - \pi/6}$ **16.** $\displaystyle\lim_{x\to -a}\frac{(1/x^2) - (1/a^2)}{x+a}$

In Exercises 17–24, express the derivatives of the given functions in terms of the derivatives f' and g' of the differentiable functions f and g.

17. $f(3 - x^2)$ **18.** $[f(\sqrt{x})]^2$

19. $f(2x)\sqrt{g(x/2)}$ **20.** $\dfrac{f(x) - g(x)}{f(x) + g(x)}$

21. $f(x + (g(x))^2)$ **22.** $f\left(\dfrac{g(x^2)}{x}\right)$

23. $f(\sin x)\, g(\cos x)$ **24.** $\sqrt{\dfrac{\cos f(x)}{\sin g(x)}}$

25. Find the tangent to the curve $x^3y + 2xy^3 = 12$ at the point $(2, 1)$.

26. Find the slope of the curve $3\sqrt{2}x\sin(\pi y) + 8y\cos(\pi x) = 2$ at the point $\left(\frac{1}{3}, \frac{1}{4}\right)$.

Find the indefinite integrals in Exercises 27–30.

27. $\displaystyle\int \frac{1+x^4}{x^2}\,dx$ **28.** $\displaystyle\int \frac{1+x}{\sqrt{x}}\,dx$

29. $\displaystyle\int \frac{2 + 3\sin x}{\cos^2 x}\,dx$ **30.** $\displaystyle\int (2x+1)^4\,dx$

31. Find $f(x)$ given that $f'(x) = 12x^2 + 12x^3$ and $f(1) = 0$.

32. Find $g(x)$ if $g'(x) = \sin(x/3) + \cos(x/6)$ and the graph of g passes through the point $(\pi, 2)$.

33. Differentiate $x\sin x + \cos x$ and $x\cos x - \sin x$, and use the results to find the indefinite integrals

$$I_1 = \int x\cos x\,dx \quad\text{and}\quad I_2 = \int x\sin x\,dx.$$

34. Suppose that $f'(x) = f(x)$ for every x, and let $g(x) = x\,f(x)$. Calculate the first several derivatives of g and guess a formula for the nth-order derivative $g^{(n)}(x)$. Verify your guess by induction.

35. Find an equation of the straight line that passes through the origin and is tangent to the curve $y = x^3 + 2$.

36. Find an equation of the straight lines that pass through the point $(0, 1)$ and are tangent to the curve $y = \sqrt{2 + x^2}$.

37. Show that $\dfrac{d}{dx}\big(\sin^n x\sin(nx)\big) = n\sin^{n-1}x\sin((n+1)x)$. At what points x in $[0, \pi]$ does the graph of $y = \sin^n x\sin(nx)$ have a horizontal tangent. Assume that $n \ge 2$.

38. Find differentiation formulas for $y = \sin^n x\cos(nx)$, $y = \cos^n x\sin(nx)$, and $y = \cos^n x\cos(nx)$ analogous to the one given for $y = \sin^n x\sin(nx)$ in the previous exercise.

39. Let Q be the point $(0, 1)$. Find all points P on the curve $y = x^2$ such that the line PQ is normal to $y = x^2$ at P. What is the shortest distance from Q to the curve $y = x^2$?

40. **(Average and marginal profit)** Figure 2.41 shows the graph of the profit $\$P(x)$ realized by a grain exporter from its sale of x tonnes of wheat. Thus, the average profit per tonne is $\$P(x)/x$. Show that the maximum average profit occurs when the average profit equals the marginal profit. What is the geometric significance of this fact in the figure?

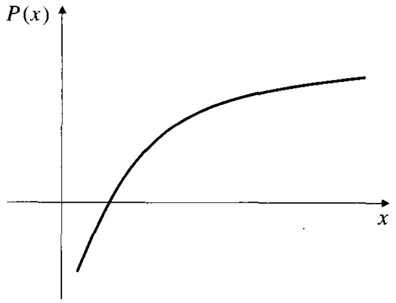

Figure 2.41

41. **(Gravitational attraction)** The gravitational attraction of the earth on a mass m at distance r from the centre of the earth is a continuous function $F(r)$ given for $r \geq 0$ by

$$F(r) = \begin{cases} \dfrac{mgR^2}{r^2} & \text{if } r \geq R \\ mkr & \text{if } 0 \leq r < R \end{cases}$$

where R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth.

(a) Find the constant k in terms of g and R.

(b) F decreases as m moves away from the surface of the earth, either upward or downward. Show that F decreases as r increases from R at twice the rate at which F decreases as r decreases from R.

42. **(Compressibility of a gas)** The isothermal compressibility of a gas is the relative rate of change of the volume V with respect to the pressure P, at a constant temperature T, that is,

$$\frac{1}{V}\frac{dV}{dP}.$$

For a sample of an ideal gas, the temperature, pressure, and volume satisfy the equation $PV = kT$, where k is a constant related to the number of molecules of gas present in the sample. Show that the isothermal compressibility of such a gas is the negative reciprocal of the pressure:

$$\frac{1}{V}\frac{dV}{dP} = -\frac{1}{P}.$$

43. A ball is thrown upward with an initial speed of 10 m/s from the top of a building. A second ball is thrown upward with an initial speed of 20 m/s from the ground. Both balls achieve the same maximum height above the ground. How tall is the building?

44. A ball is dropped from the top of a 60 m high tower at the same instant that a second ball is thrown upward from the ground at the base of the tower. The balls collide at a height of 30 m above the ground. With what initial velocity was the second ball thrown? How fast is each ball moving when they collide?

45. **(Braking distance)** A car's brakes can decelerate the car at 20 ft/s^2. How fast can the car travel if it must be able to stop in a distance of 160 ft?

46. **(Measuring variations in g)** The period P of a pendulum of length L is given by

$$P = 2\pi\sqrt{L/g},$$

where g is the acceleration of gravity.

(a) Assuming that L remains fixed, show that a 1% increase in g results in approximately a 1/2% decrease in the period P. (Variations in the period of a pendulum can be used to detect small variations in g from place to place on the earth's surface.)

(b) For fixed g, what percentage change in L will produce a 1% increase in P?

Challenging Problems

1. René Descartes, the inventor of analytic geometry, calculated the tangent to a parabola (or a circle or other quadratic curve) at a given point (x_0, y_0) on the curve by looking for a straight line through (x_0, y_0) having only one intersection with the given curve. Illustrate his method by writing the equation of a line through (a, a^2), having arbitrary slope m, and then finding the value of m for which the line has only one intersection with the parabola $y = x^2$. Why does the method not work for more general curves?

2. Given that $f'(x) = 1/x$ and $f(2) = 9$, find:

(a) $\displaystyle\lim_{x\to 2}\frac{f(x^2+5) - f(9)}{x-2}$ (b) $\displaystyle\lim_{x\to 2}\frac{\sqrt{f(x)} - 3}{x-2}$

3. Suppose that $f'(4) = 3$, $g'(4) = 7$, $g(4) = 4$, and $g(x) \neq 4$ for $x \neq 4$. Find:

(a) $\displaystyle\lim_{x\to 4}\big(f(x) - f(4)\big)$ (b) $\displaystyle\lim_{x\to 4}\frac{f(x) - f(4)}{x^2 - 16}$

(c) $\displaystyle\lim_{x\to 4}\frac{f(x) - f(4)}{\sqrt{x} - 2}$ (d) $\displaystyle\lim_{x\to 4}\frac{f(x) - f(4)}{(1/x) - (1/4)}$

(e) $\displaystyle\lim_{x\to 4}\frac{f(x) - f(4)}{g(x) - 4}$ (f) $\displaystyle\lim_{x\to 4}\frac{f(g(x)) - f(4)}{x-4}$

4. Let $f(x) = \begin{cases} x & \text{if } x = 1,\ 1/2,\ 1/3,\ 1/4,\ \ldots \\ x^2 & \text{otherwise.} \end{cases}$

(a) Find all points at which f is continuous. In particular, is it continuous at $x = 0$?

(b) Is the following statement true or false? Justify your answer. For any two real numbers a and b, there is some x between a and b such that $f(x) = (f(a) + f(b))/2$.

(c) Find all points at which f is differentiable. In particular, is it differentiable at $x = 0$?

5. Suppose $f(0) = 0$ and $|f(x)| > \sqrt{|x|}$ for all x. Show that $f'(0)$ does not exist.

6. Suppose that f is a function satisfying the following conditions: $f'(0) = k$, $f(0) \neq 0$, and $f(x+y) = f(x)f(y)$ for

all x and y. Show that $f(0) = 1$ and that $f'(x) = k\,f(x)$ for every x. (We will study functions with these properties in Chapter 3.)

7. Suppose the function g satisfies the conditions: $g'(0) = k$, and $g(x + y) = g(x) + g(y)$ for all x and y. Show that:

(a) $g(0) = 0$, (b) $g'(x) = k$ for all x, and

(c) $g(x) = kx$ for all x. *Hint:* let $h(x) = g(x) - g'(0)x$.

8. (a) If f is differentiable at x, show that

(i) $\displaystyle\lim_{h\to 0} \frac{f(x) - f(x-h)}{h} = f'(x)$

(ii) $\displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x)$

(b) Show that the existence of the limit in (i) guarantees that f is differentiable at x.

(c) Show that the existence of the limit in (ii) does *not* guarantee that f is differentiable at x. *Hint:* consider the function $f(x) = |x|$ at $x = 0$.

9. Show that there is a line through $(a, 0)$ that is tangent to the curve $y = x^3$ at $x = 3a/2$. If $a \neq 0$, is there any other line through $(a, 0)$ that is tangent to the curve? If (x_0, y_0) is an arbitrary point, what is the maximum number of lines through (x_0, y_0) that can be tangent to $y = x^3$? the minimum number?

10. Make a sketch showing that there are two straight lines, each of which is tangent to both of the parabolas $y = x^2 + 4x + 1$ and $y = -x^2 + 4x - 1$. Find equations of the two lines.

11. Show that if $b > 1/2$, there are three straight lines through $(0, b)$, each of which is normal to the curve $y = x^2$. How many such lines are there if $b = 1/2$? if $b < 1/2$?

12. **(Distance from a point to a curve)** Find the point on the curve $y = x^2$ that is closest to the point $(3, 0)$. *Hint:* the line from $(3, 0)$ to the closest point Q on the parabola is normal to the parabola at Q.

$*$ **13.** **(Envelope of a family of lines)** Show that for each value of the parameter m, the line $y = mx - (m^2/4)$ is tangent to the parabola $y = x^2$. (The parabola is called the *envelope* of the family of lines $y = mx - (m^2/4)$.) Find $f(m)$ such that the family of lines $y = mx + f(m)$ has envelope the parabola $y = Ax^2 + Bx + C$.

$*$ **14.** **(Common tangents)** Consider the two parabolas with equations $y = x^2$ and $y = Ax^2 + Bx + C$. We assume that $A \neq 0$, and if $A = 1$, then either $B \neq 0$ or $C \neq 0$, so that the two equations do represent different parabolas. Show that:

(a) the two parabolas are tangent to each other if $B^2 = 4C(A - 1)$;

(b) the parabolas have two common tangent lines if and only if $A \neq 1$ and $A\big(B^2 - 4C(A-1)\big) > 0$;

(c) the parabolas have exactly one common tangent line if either $A = 1$ and $B \neq 0$, or $A \neq 1$ and $B^2 = 4C(A-1)$;

(d) the parabolas have no common tangent lines if either $A = 1$ and $B = 0$, or $A \neq 1$ and $A\big(B^2 - 4C(A-1)\big) < 0$.

Make sketches illustrating each of the above possibilities.

15. Let C be the graph of $y = x^3$.

(a) Show that if $a \neq 0$ then the tangent to C at $x = a$ also intersects C at a second point $x = b$.

(b) Show that the slope of C at $x = b$ is four times its slope at $x = a$.

(c) Can any line be tangent to C at more than one point?

(d) Can any line be tangent to the graph of $y = Ax^3 + Bx^2 + Cx + D$ at more than one point?

$*$ **16.** Let C be the graph of $y = x^4 - 2x^2$.

(a) Find all horizontal lines that are tangent to C.

(b) One of the lines found in (a) is tangent to C at two different points. Show that there are no other lines that have this property.

(c) Find an equation of a straight line that is tangent to the graph of $y = x^4 - 2x^2 + x$ at two different points. Can there exist more than one such line? Why?

17. **(Double tangents)** A line tangent to the quartic (fourth-degree polynomial) curve C with equation $y = ax^4 + bx^3 + cx^2 + dx + e$ at $x = p$ may intersect C at zero, one, or two other points. If it meets C at only one other point $x = q$, it must be tangent to C at that point also, and it is thus a "double tangent."

(a) Find the condition that must be satisfied by the coefficients of the quartic to ensure that there does exist such a double tangent, and show that there cannot be more than one such double tangent. Illustrate this by applying your results to $y = x^4 - 2x^2 + x - 1$.

(b) If the line PQ is tangent to C at two distinct points $x = p$ and $x = q$, show that PQ is parallel to the line tangent to C at $x = (p+q)/2$.

(c) If the line PQ is tangent to C at two distinct points $x = p$ and $x = q$, show that C has two distinct inflection points R and S and that RS is parallel to PQ.

18. Verify the following formulas for every positive integer n:

(a) $\displaystyle\frac{d^n}{dx^n}\cos(ax) = a^n \cos\left(ax + \frac{n\pi}{2}\right)$

(b) $\displaystyle\frac{d^n}{dx^n}\sin(ax) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$

(c) $\displaystyle\frac{d^n}{dx^n}\left(\cos^4 x + \sin^4 x\right) = 4^{n-1}\cos\left(4x + \frac{n\pi}{2}\right)$

19. **(Rocket with a parachute)** A rocket is fired from the top of a tower at time $t = 0$. It experiences constant upward acceleration until its fuel is used up. Thereafter its acceleration is the constant downward acceleration of gravity until, during its fall, it deploys a parachute that gives it a constant upward acceleration again to slow it down. The rocket hits the ground near the base of the tower. The upward velocity v (in metres per second) is graphed against time in Figure 2.42. From information in the figure answer the following questions:

(a) How long did the fuel last?

(b) When was the rocket's height maximum?

(c) When was the parachute deployed?

(d) What was the rocket's upward acceleration while its motor was firing?

(e) What was the maximum height achieved by the rocket?

(f) How high was the tower from which the rocket was fired?

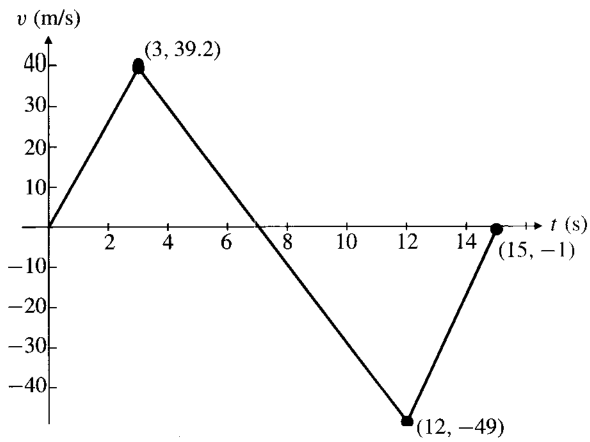

Figure 2.42