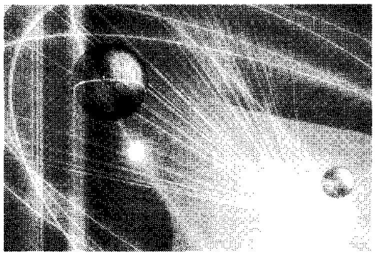


Appendix I

Complex Numbers

Many of the problems to which mathematics is applied involve the solution of equations. Over the centuries the number system had to be expanded many times to provide solutions for more and more kinds of equations. The natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

are inadequate for the solutions of equations of the form

$$x + n = m, \qquad (m, n \in \mathbb{N}).$$

Zero and negative numbers can be added to create the integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

in which that equation has the solution $x = m - n$ even if $m < n$. (Historically, this extension of the number system came much later than some of those mentioned below.) Some equations of the form

$$nx = m, \qquad (m, n \in \mathbb{Z}, \quad n \neq 0)$$

cannot be solved in the integers. Another extension is made to to include numbers of the form m/n, thus producing the set of rational numbers

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, \quad n \neq 0 \right\}.$$

Every linear equation

$$ax = b, \qquad (a, b \in \mathbb{Q}, \quad a \neq 0)$$

has a solution $x = b/a$ in $\mathbb{Q}$, but the quadratic equation

$$x^2 = 2$$

has no solution in $\mathbb{Q}$, as was shown in Section P.1. Another extension enriches the rational numbers to the real numbers $\mathbb{R}$ in which some equations like $x^2 = 2$ have solutions. However, other quadratic equations, for instance,

$$x^2 = -1$$

do not have solutions, even in the real numbers, so the extension process is not complete. In order to be able to solve any quadratic equation, we need to extend the real number system to a larger set, which we call **the complex number system**. In this appendix we will define complex numbers and develop some of their basic properties.

Definition of Complex Numbers

We begin by defining the symbol i, called **the imaginary unit**[1], to have the property

$$i^2 = -1.$$

Thus, we could also call i the **square root of** -1 and denote it $\sqrt{-1}$. Of course, i is not a real number; no real number has a negative square.

DEFINITION 1

> A **complex number** is an expression of the form
>
> $$a + bi \qquad \text{or} \qquad a + ib$$
>
> where a and b are *real numbers*, and i is the imaginary unit.

For example, $3 + 2i$, $\frac{7}{2} - \frac{2}{3}i$, $i\pi = 0 + i\pi$, and $-3 = -3 + 0i$ are all complex numbers. The last of these examples shows that every real number can be regarded as a complex number. (We will normally use $a + bi$ unless b is a complicated expression, in which case we will write $a + ib$ instead. Either form is acceptable.)

It is often convenient to represent a complex number by a single letter; w and z are frequently used for this purpose. If a, b, x, and y are real numbers, and

$$w = a + bi \qquad \text{and} \qquad z = x + yi,$$

then we can refer to the complex numbers w and z. Note that $w = z$ if and only if $a = x$ and $b = y$. Of special importance are the complex numbers

$$0 = 0 + 0i, \qquad 1 = 1 + 0i, \qquad \text{and} \qquad i = 0 + 1i.$$

DEFINITION 2

> If $z = x + yi$ is a complex number (where x and y are real), we call x the **real part** of z and denote it $\operatorname{Re}(z)$. We call y the **imaginary part** of z and denote it $\operatorname{Im}(z)$:
>
> $$\operatorname{Re}(z) = \operatorname{Re}(x + yi) = x, \qquad \operatorname{Im}(z) = \operatorname{Im}(x + yi) = y.$$

Note that both the real and imaginary parts of a complex number are real numbers.

$$\begin{aligned} &\operatorname{Re}(3 - 5i) = 3 && \operatorname{Im}(3 - 5i) = -5 \\ &\operatorname{Re}(2i) = \operatorname{Re}(0 + 2i) = 0 && \operatorname{Im}(2i) = \operatorname{Im}(0 + 2i) = 2 \\ &\operatorname{Re}(-7) = \operatorname{Re}(-7 + 0i) = -7 && \operatorname{Im}(-7) = \operatorname{Im}(-7 + 0i) = 0. \end{aligned}$$

Graphical Representation of Complex Numbers

Since complex numbers are constructed from pairs of real numbers (their real and imaginary parts), it is natural to represent complex numbers graphically as points in a Cartesian plane. We use the point with coordinates (a, b) to represent the complex number $w = a + ib$. In particular, the origin $(0, 0)$ represents the complex number 0, the point $(1, 0)$ represents the complex number $1 = 1 + 0i$, and the point $(0, 1)$ represents the point $i = 0 + 1i$. (See Figure I.1.)

[1] In some fields, for example, electrical engineering, the imaginary unit is denoted j instead of i. Like "negative," "surd," and "irrational," the term "imaginary" suggests the distrust that greeted the new kinds of numbers when they were first introduced.

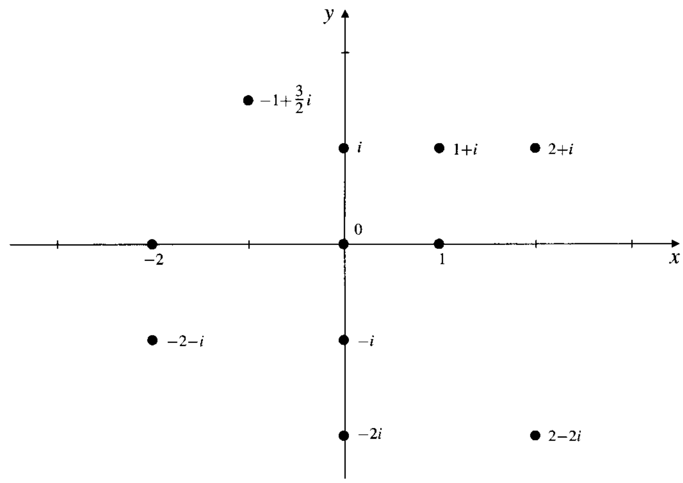


Figure I.1 An Argand diagram representing the complex plane

Such a representation of complex numbers as points in a plane is called an **Argand diagram**. Since each complex number is represented by a unique point in the plane, the set of all complex numbers is often referred to as **the complex plane**. The symbol $\mathbb{C}$ is used to represent the set of all complex numbers and, equivalently, the complex plane:

$$\mathbb{C} = \{x + yi \ : \ x,\ y,\ \in\ \mathbb{R}\}\ .$$

The points on the x-axis of the complex plane correspond to real numbers ($x = x + 0i$), so the x-axis is called the **real axis**. The points on the y-axis correspond to **pure imaginary** numbers ($yi = 0 + yi$), so the y-axis is called the **imaginary axis**.

It can be helpful to use the *polar coordinates* of a point in the complex plane.

DEFINITION 3

> The distance from the origin to the point (a, b) corresponding to the complex number $w = a + bi$ is called the **modulus** of w and is denoted by $|w|$ or $|a + bi|$:
>
> $$|w| = |a + bi| = \sqrt{a^2 + b^2}.$$
>
> If the line from the origin to (a, b) makes angle θ with the positive direction of the real axis (with positive angles measured counterclockwise), then we call θ an **argument** of the complex number $w = a + bi$ and denote it by $\arg(w)$ or $\arg(a + bi)$. (See Figure I.2.)

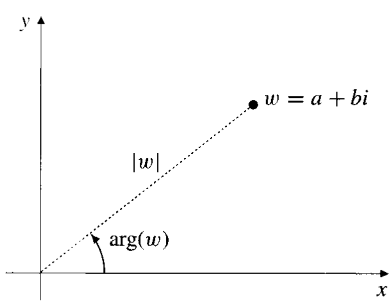


Figure I.2 The modulus and argument of a complex number

The modulus of a complex number is always real and nonnegative. It is positive unless the complex number is 0. Modulus plays a similar role for complex numbers that absolute value does for real numbers. Indeed, sometimes modulus is called absolute value.

Arguments of complex numbers are not unique. If $w = a + bi \neq 0$, then any two possible values for $\arg(w)$ differ by an integer multiple of 2π. The symbol $\arg(w)$ actually represents not a single number, but a set of numbers. When we write $\arg(w) = \theta$, we are saying that the set $\arg(w)$ contains all numbers of the form $\theta + 2k\pi$, where k is an integer. Similarly, the statement $\arg(z) = \arg(w)$ says that two sets are identical.

If $w = a + bi$, where $a = \mathrm{Re}\,(w) \neq 0$, then

$$\tan \arg(w) = \tan \arg(a + bi) = \frac{b}{a}.$$

This means that $\tan\theta = b/a$ for every θ in the set $\arg(w)$.

It is sometimes convenient to restrict $\theta = \arg(w)$ to an interval of length 2π, say, the interval $0 \leq \theta < 2\pi$, or $-\pi < \theta \leq \pi$, so that nonzero complex numbers will have unique arguments. We will call the value of $\arg(w)$ in the interval $-\pi < \theta \leq \pi$ the **principal argument** of w and denote it $\mathrm{Arg}\,(w)$. Every complex number w except 0 has a unique principal argument $\mathrm{Arg}\,(w)$.

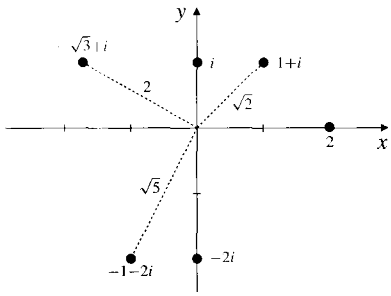


Figure I.3 Some complex numbers with their moduli

Example 1 **(Some moduli and principal arguments)** See Figure I.3.

$$\begin{aligned}
&|2| = 2 && \mathrm{Arg}\,(2) = 0\\
&|1+i| = \sqrt{2} && \mathrm{Arg}\,(1+i) = \pi/4\\
&|i| = 1 && \mathrm{Arg}\,(i) = \pi/2\\
&|-2i| = 2 && \mathrm{Arg}\,(-2i) = -\pi/2\\
&|-\sqrt{3}+i| = 2 && \mathrm{Arg}\,(-\sqrt{3}+i) = 5\pi/6\\
&|-1-2i| = \sqrt{5} && \mathrm{Arg}\,(-1-2i) = -\pi + \tan^{-1}(2).
\end{aligned}$$

Remark If $z = x + yi$ and $\mathrm{Re}\,(z) = x > 0$, then $\mathrm{Arg}\,(z) = \tan^{-1}(y/x)$. Many computer spreadsheets implement a two-variable arctan function denoted $\mathrm{atan2}(x, y)$ which gives the polar angle of (x, y) in the interval $]-\pi, \pi]$. Thus

$$\mathrm{Arg}\,(x + yi) = \mathrm{atan2}(x, y).$$

Given the modulus $r = |w|$ and any value of the argument $\theta = \arg(w)$ of a complex number $w = a + bi$, we have $a = r\cos\theta$ and $b = r\sin\theta$, so w can be expressed in terms of its modulus and argument as

$$w = r\cos\theta + i\,r\sin\theta.$$

The expression on the right side is called the **polar representation** of w.

DEFINITION 4

The **conjugate** or **complex conjugate** of a complex number $w = a + bi$ is another complex number, denoted $\overline{w}$, given by

$$\overline{w} = a - bi.$$

Example 2 $\overline{2 - 3i} = 2 + 3i, \quad \overline{3} = 3, \quad \overline{2i} = -2i.$

Observe that

$$\begin{aligned}
&\mathrm{Re}\,(\overline{w}) = \mathrm{Re}\,(w) && |\overline{w}| = |w|\\
&\mathrm{Im}\,(\overline{w}) = -\mathrm{Im}\,(w) && \arg(\overline{w}) = -\arg(w).
\end{aligned}$$

In an Argand diagram the point $\overline{w}$ is the reflection of the point w in the real axis. (See Figure I.4.)

Note that w is real ($\text{Im}(w) = 0$) if and only if $\overline{w} = w$. Also, w is pure imaginary ($\text{Re}(w) = 0$) if and only if $\overline{w} = -w$. (Here, $-w = -a - bi$ if $w = a + bi$.)

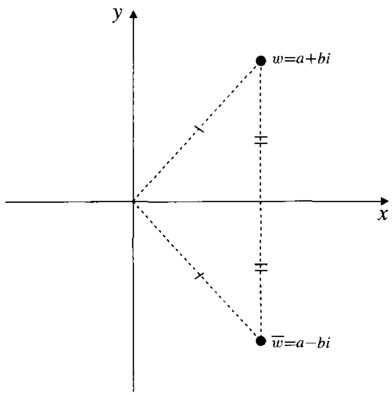


Figure I.4 A complex number and its conjugate are mirror images of each other in the real axis

Complex Arithmetic

Like real numbers, complex numbers can be added, subtracted, multiplied, and divided. Two complex numbers are added or subtracted as though they are two-dimensional vectors whose components are their real and imaginary parts.

The sum and difference of complex numbers

If $w = a + bi$ and $z = x + yi$, where a, b, x, and y are real numbers, then

$$w + z = (a + x) + (b + y)i$$
$$w - z = (a - x) + (b - y)i.$$

In an Argand diagram the points $w + z$ and $w - z$ are the points whose position vectors are, respectively, the sum and difference of the position vectors of the points w and z. (See Figure I.5.) In particular, the complex number $a + bi$ is the sum of the real number $a = a + 0i$ and the pure imaginary number $bi = 0 + bi$.

Complex addition obeys the same rules as real addition: if w_1, w_2, and w_3 are three complex numbers, the following are easily verified:

$w_1 + w_2 = w_2 + w_1$	Addition is commutative.
$(w_1 + w_2) + w_3 = w_1 + (w_2 + w_3)$	Addition is associative.
$\lvert w_1 \pm w_2 \rvert \le \lvert w_1 \rvert + \lvert w_2 \rvert$	the triangle inequality

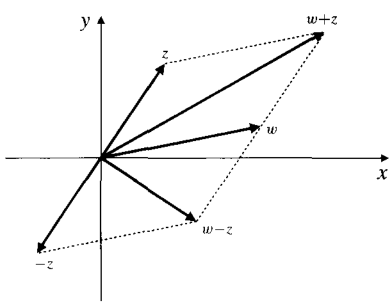


Figure I.5 Complex numbers are added and subtracted vectorially. Observe the parallelograms

Note that $|w_1 - w_2|$ is the distance between the two points w_1 and w_2 in the complex plane. Thus, the triangle inequality says that in the triangle with vertices w_1, $\mp w_2$ and 0, the length of one side is less than the sum of the other two.

It is also easily verified that the conjugate of a sum (or difference) is the sum (or difference) of the conjugates:

$$\overline{w + z} = \overline{w} + \overline{z}.$$

Example 3

(a) If $w = 2 + 3i$ and $z = 4 - 5i$, then

$$w + z = (2 + 4) + (3 - 5)i = 6 - 2i$$
$$w - z = (2 - 4) + (3 - (-5))i = -2 + 8i.$$

(b) $3i + (1 - 2i) - (2 + 3i) + 5 = 4 - 2i$. ■

Multiplication of the complex numbers $w = a + bi$ and $z = x + yi$ is carried out by formally multiplying the binomial expressions and replacing i^2 by -1:

$$wz = (a + bi)(x + yi) = ax + ayi + bxi + byi^2$$
$$= (ax - by) + (ay + bx)i.$$

The product of complex numbers

If $w = a + bi$ and $z = x + yi$, where a, b, x, and y are real numbers, then

$$wz = (ax - by) + (ay + bx)i.$$

Example 4

(a) $(2 + 3i)(1 - 2i) = 2 - 4i + 3i - 6i^2 = 8 - i$.

(b) $i(5 - 4i) = 5i - 4i^2 = 4 + 5i$.

(c) $(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2$.

Part (c) of the example above shows that the square of the modulus of a complex number is the product of that number with its complex conjugate:

$$w\,\overline{w} = |w|^2.$$

Complex multiplication has many properties in common with real multiplication. In particular, if w_1, w_2, and w_3 are complex numbers, then

$w_1w_2 = w_2w_1$	Multiplication is commutative.
$(w_1w_2)w_3 = w_1(w_2w_3)$	Multiplication is associative.
$w_1(w_2 + w_3) = w_1w_2 + w_1w_3$	Multiplication distributes over addition.

The conjugate of a product is the product of the conjugates:

$$\overline{wz} = \overline{w}\,\overline{z}.$$

To see this, let $w = a + bi$ and $z = x + yi$. Then

$$\begin{aligned}\overline{wz} &= \overline{(ax - by) + (ay + bx)i}\\ &= (ax - by) - (ay + bx)i\\ &= (a - bi)(x - yi) = \overline{w}\,\overline{z}.\end{aligned}$$

It is particularly easy to determine the product of complex numbers expressed in polar form. If

$$w = r(\cos\theta + i\sin\theta) \qquad \text{and} \qquad z = s(\cos\phi + i\sin\phi),$$

where $r = |w|$, $\theta = \arg(w)$, $s = |z|$, and $\phi = \arg(z)$, then

$$\begin{aligned}wz &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= rs\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\big)\\ &= rs\big(\cos(\theta + \phi) + i\sin(\theta + \phi)\big).\end{aligned}$$

(See Figure I.6.) Since arguments are only determined up to integer multiples of 2π, we have proved that

The modulus and argument of a product

$$|wz| = |w||z| \qquad \text{and} \qquad \arg(wz) = \arg(w) + \arg(z).$$

The second of these equations says that the set $\arg(wz)$ consists of all numbers $\theta + \phi$, where θ belongs to the set $\arg(w)$ and ϕ to the set $\arg(z)$.

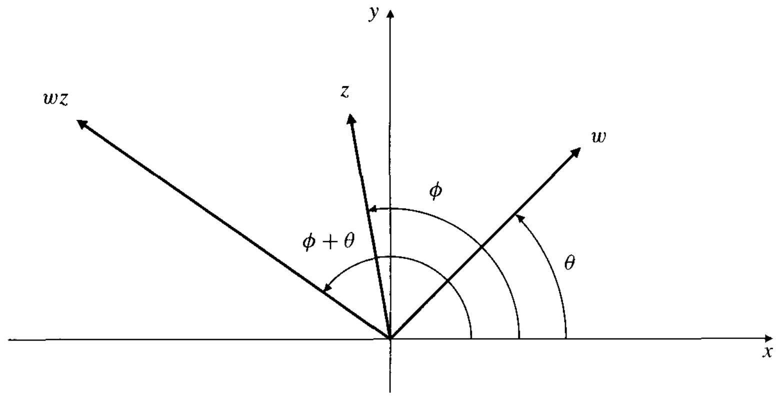


Figure I.6 The argument of a product is the sum of the arguments of the factors

More generally, if $w_1,\ w_2,\ \ldots\ w_n$ are complex numbers, then

$$\begin{aligned}|w_1 w_2 \cdots w_n| &= |w_1||w_2|\cdots|w_n| \\ \arg(w_1 w_2 \cdots w_n) &= \arg(w_1) + \arg(w_2) + \cdots + \arg(w_n).\end{aligned}$$

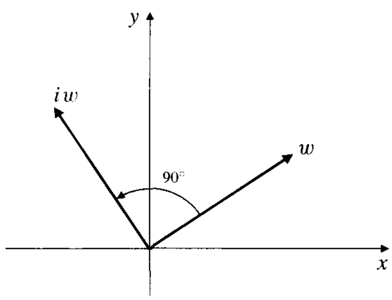


Figure I.7 Multiplication by i corresponds to counterclockwise rotation by 90°

Multiplication of a complex number by i has a particularly simple geometric interpretation in an Argand diagram. Since $|i| = 1$ and $\arg(i) = \pi/2$, multiplication of $w = a + bi$ by i leaves the modulus of w, unchanged but increases its argument by $\pi/2$. (See Figure I.7.) Thus, multiplication by i rotates the position vector of w counterclockwise by 90° about the origin.

Let $z = \cos\theta + i\sin\theta$. Then $|z| = 1$ and $\arg(z) = \theta$. Since the modulus of a product is the product of the moduli of the factors and the argument of a product is the sum of the arguments of the factors, we have $|z^n| = |z|^n = 1$ and $\arg(z^n) = n\arg(z) = n\theta$. Thus,

$$z^n = \cos n\theta + i\sin n\theta,$$

and we have proved

THEOREM 1

de Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Remark The study of complex-valued functions of a complex variable is beyond the scope of this book. However, we point out that there is a complex version of the exponential function having the following property: if $z = x + iy$ (where x and y are real), then

$$e^z = e^{x+iy} = e^x\, e^{iy} = e^x(\cos y + i\sin y).$$

Thus the modulus of e^z is $e^{\mathrm{Re}\,(z)}$ and $\mathrm{Im}\,(z)$ is a value of $\arg(e^z)$. In this context, de Moivre's Theorem just says

$$(e^{i\theta})^n = e^{in\theta}.$$

Example 5 Express $(1+i)^5$ in the form $a+bi$.

Solution Since $|(1+i)^5| = |1+i|^5 = \left(\sqrt{2}\right)^5 = 4\sqrt{2}$, and $\arg\left((1+i)^5\right) = 5\arg(1+i) = \frac{5\pi}{4}$, we have

$$(1+i)^5 = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -4-4i.$$

de Moivre's Theorem can be used to generate trigonometric identities for multiples of an angle. For example, for $n=2$ we have

$$\cos 2\theta + i\sin 2\theta = \left(\cos\theta + i\sin\theta\right)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.$$

Thus, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, and $\sin 2\theta = 2\sin\theta\cos\theta$.

The **reciprocal** of the nonzero complex number $w = a+bi$ can be calculated by multiplying the numerator and denominator of the reciprocal expression by the conjugate of w:

$$w^{-1} = \frac{1}{w} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = \frac{\overline{w}}{|w|^2}.$$

Since $|\overline{w}| = |w|$, and $\arg(\overline{w}) = -\arg(w)$, we have

$$\left|\frac{1}{w}\right| = \frac{|\overline{w}|}{|w|^2} = \frac{1}{|w|} \quad \text{and} \quad \arg\left(\frac{1}{w}\right) = -\arg(w).$$

The **quotient** z/w of two complex numbers $z = x+yi$ and $w = a+bi$ is the product of z and $1/w$, so

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} = \frac{(x+yi)(a-bi)}{a^2+b^2} = \frac{xa+yb+i(ya-xb)}{a^2+b^2}.$$

We have

The modulus and argument of a quotient

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \quad \text{and} \quad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w).$$

The set $\arg(z/w)$ consists of all numbers $\theta - \phi$ where θ belongs to the set $\arg(z)$ and ϕ to the set $\arg(w)$.

Example 6 Simplify (a) $\dfrac{2+3i}{4-i}$ and (b) $\dfrac{i}{1+i\sqrt{3}}$.

Solution

(a) $\dfrac{2+3i}{4-i} = \dfrac{(2+3i)(4+i)}{(4-i)(4+i)} = \dfrac{8-3+(2+12)i}{4^2+1^2} = \dfrac{5}{17} + \dfrac{14}{17}i.$

(b) $\dfrac{i}{1+i\sqrt{3}} = \dfrac{i(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \dfrac{\sqrt{3}+i}{1^2+3} = \dfrac{\sqrt{3}}{4} + \dfrac{1}{4}i.$

Alternatively, since $|1+i\sqrt{3}| = 2$ and $\arg(1+i\sqrt{3}) = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$, the quotient in (b) has modulus $\dfrac{1}{2}$ and argument $\dfrac{\pi}{2} - \dfrac{\pi}{3} = \dfrac{\pi}{6}$. Thus

$$\frac{i}{1+i\sqrt{3}} = \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{4} + \frac{1}{4}i.$$

Roots of Complex Numbers

If a is a positive real number, there are two distinct real numbers whose square is a. These are usually denoted

$\sqrt{a}$ (the positive square root of a and)

$-\sqrt{a}$ (the negative square root of a).

Every nonzero complex number $z = x + yi$ (where $x^2 + y^2 > 0$) also has two square roots; if w_1 is a complex number such that $w_1^2 = z$, then $w_2 = -w_1$ also satisfies $w_2^2 = z$. Again, we would like to single out one of these roots and call it $\sqrt{z}$.

Let $r = |z|$, so that $r > 0$. Let $\theta = \operatorname{Arg}(z)$. Thus $-\pi < \theta \le \pi$. Since

$$z = r\big(\cos\theta + i\sin\theta\big),$$

the complex number

$$w = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

clearly satisfies $w^2 = z$. We call this w the **principal square root** of z and denote it $\sqrt{z}$. The two solutions of the equation $w^2 = z$ are, thus, $w = \sqrt{z}$ and $w = -\sqrt{z}$. Observe that the real part of $\sqrt{z}$ is always nonnegative since $\cos(\theta/2) \ge 0$ for $-\pi/2 < \theta \le \pi/2$. In this interval $\sin(\theta/2) = 0$ only if $\theta = 0$ in which case $\sqrt{z}$ is real and positive.

Example 7

(a) $\sqrt{4} = \sqrt{4(\cos 0 + i\sin 0)} = 2.$

(b) $\sqrt{i} = \sqrt{1\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)} = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i.$

(c) $\sqrt{-4i} = \sqrt{4\left[\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right]} = 2\left[\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right]$
$= \sqrt{2} - i\sqrt{2}.$

(d) $\sqrt{-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}} = \sqrt{\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i.$

Given a nonzero complex number z we can find n distinct complex numbers w that satisfy $w^n = z$. These n numbers are called nth roots of z. For example, if $z = 1 = \cos 0 + i \sin 0$, then each of the numbers

$$n \qquad n$$

$$w_4 = \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n}$$

$$\vdots$$

$$w_n = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$$

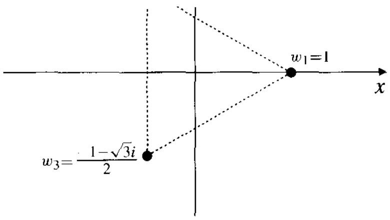


Figure I.8 The cube roots of unity

satisfies $w^n = 1$ so is an nth root of 1. (These numbers are usually called the nth roots of unity.) Figure I.8 shows the three cube roots of 1. Observe that they are at the three vertices of an equilateral triangle with centre at the origin and one vertex at 1. In general, the n nth roots of unity lie on a circle of radius 1 centred at the origin, at the vertices of a regular n-sided polygon with one vertex at 1.

If z is any nonzero complex number, and θ is the principal argument of z $(-\pi < \theta \le \pi)$, then the number

$$w_1 = |z|^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

is called the **principal** nth root of z. All the nth roots of z are on the circle of radius $|z|^{1/n}$ centred at the origin and are at the vertices of a regular n-sided polygon with one vertex at w_1. (See Figure I.9.) The other nth roots are

$$w_2 = |z|^{1/n}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right)$$

$$w_3 = |z|^{1/n}\left(\cos\frac{\theta+4\pi}{n} + i\sin\frac{\theta+4\pi}{n}\right)$$

$$\vdots$$

$$w_n = |z|^{1/n}\left(\cos\frac{\theta+2(n-1)\pi}{n} + i\sin\frac{\theta+2(n-1)\pi}{n}\right).$$

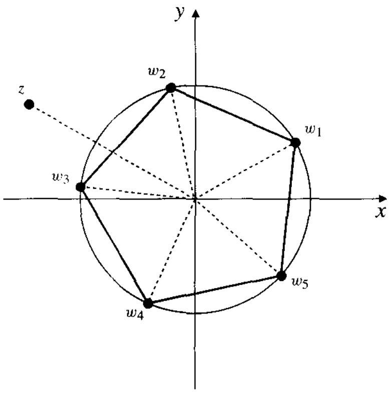


Figure I.9 The five 5th roots of z

We can obtain all n of the nth roots of z by multiplying the principal nth root by the nth roots of unity.

Example 8 Find the 4th roots of -4. Sketch them in an Argand diagram.

Solution Since $|-4|^{1/4} = \sqrt{2}$ and $\arg(-4) = \pi$, the principal 4th root of -4 is

$$w_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1 + i.$$

The other three 4th roots are at the vertices of a square with centre at the origin and one vertex at $1 + i$. (See Figure I.10.) Thus the other roots are

$$w_2 = -1 + i, \qquad w_3 = -1 - i, \qquad w_4 = 1 - i.$$

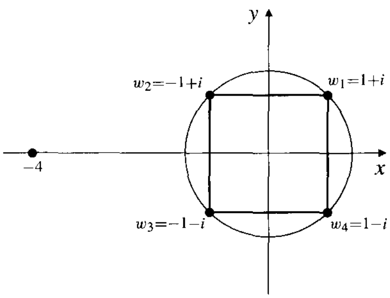


Figure I.10 The four 4th roots of -4

Exercises

In Exercises 1–4, find the real and imaginary parts (Re (z) and Im (z)) of the given complex numbers z, and sketch the position of each number in the complex plane (i.e., in an Argand diagram).

1. $z = -5 + 2i$ **2.** $z = 4 - i$

3. $z = -\pi i$ **4.** $z = -6$

In Exercises 5–15, find the modulus $r = |z|$ and the principal argument $\theta = \operatorname{Arg}(z)$ of each given complex number z, and express z in terms of r and θ.

5. $z = -1 + i$ **6.** $z = -2$

7. $z = 3i$ **8.** $z = -5i$

9. $z = 1 + 2i$ **10.** $z = -2 + i$

11. $z = -3 - 4i$ **12.** $z = 3 - 4i$

13. $z = \sqrt{3} - i$ **14.** $z = -\sqrt{3} - 3i$

15. $z = 3\cos\dfrac{4\pi}{5} + 3i\sin\dfrac{4\pi}{5}$

16. If $\operatorname{Arg}(z) = 3\pi/4$ and $\operatorname{Arg}(w) = \pi/2$, find $\operatorname{Arg}(zw)$.

17. If $\operatorname{Arg}(z) = -5\pi/6$ and $\operatorname{Arg}(w) = \pi/4$, find $\operatorname{Arg}(z/w)$.

In Exercises 18–23, express in the form $z = x + yi$ the complex number z whose modulus and argument are given.

18. $|z| = 2$, $\arg(z) = \pi$ **19.** $|z| = 5$, $\arg(z) = \tan^{-1}\dfrac{3}{4}$

20. $|z| = 1$, $\arg(z) = \dfrac{3\pi}{4}$ **21.** $|z| = \pi$, $\arg(z) = \dfrac{\pi}{6}$

22. $|z| = 0$, $\arg(z) = 1$ **23.** $|z| = \dfrac{1}{2}$, $\arg(z) = -\dfrac{\pi}{3}$

In Exercises 24–27, find the complex conjugates of the given complex numbers.

24. $5 + 3i$ **25.** $-3 - 5i$

26. $4i$ **27.** $2 - i$

Describe geometrically (or make a sketch of) the set of points z in the complex plane satisfying the given equations or inequalities in Exercises 28–33.

28. $|z| = 2$ **29.** $|z| \le 2$

30. $|z - 2i| \le 3$ **31.** $|z - 3 + 4i| \le 5$

32. $\arg z = \dfrac{\pi}{3}$ **33.** $\pi \le \arg(z) \le \dfrac{7\pi}{4}$

Simplify the expressions in Exercises 34–43.

34. $(2 + 5i) + (3 - i)$ **35.** $i - (3 - 2i) + (7 - 3i)$

36. $(4 + i)(4 - i)$ **37.** $(1 + i)(2 - 3i)$

38. $(a + bi)(\overline{2a - bi})$ **39.** $(2 + i)^3$

40. $\dfrac{2 - i}{2 + i}$ **41.** $\dfrac{1 + 3i}{2 - i}$

42. $\dfrac{1 + i}{i(2 + 3i)}$ **43.** $\dfrac{(1 + 2i)(2 - 3i)}{(2 - i)(3 + 2i)}$

44. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.

45. Prove that $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$.

46. Express each of the complex numbers $z = 3 + i\sqrt{3}$ and $w = -1 + i\sqrt{3}$ in polar form (i.e., in terms of its modulus and argument). Use these expressions to calculate zw and z/w.

47. Repeat the previous exercise for $z = -1 + i$ and $w = 3i$.

48. Use de Moivre's Theorem to find a trigonometric identity for $\cos 3\theta$ in terms of $\cos\theta$ and one for $\sin 3\theta$ in terms of $\sin\theta$.

49. Describe the solutions, if any, of the equations (a) $\overline{z} = 2/z$ and (b) $\overline{z} = -2/z$.

50. For positive real numbers a and b it is always true that $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Does a similar identity hold for $\sqrt{zw}$, where z and w are complex numbers? *Hint:* consider $z = w = -1$.

51. Find the three cube roots of -1.

52. Find the three cube roots of $-8i$.

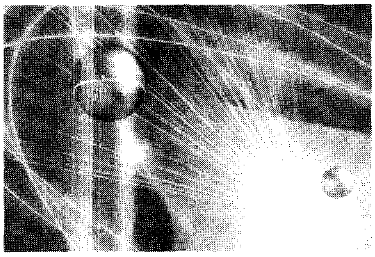
53. Find the three cube roots of $-1 + i$.

54. Find all the fourth roots of 4.

55. Find all complex solutions of the equation $z^4 + 1 - i\sqrt{3} = 0$.

56. Find all solutions of $z^5 + a^5 = 0$, where a is a positive real number.

* **57.** Show that the sum of the n nth roots of unity is zero. *Hint:* show that these roots are all powers of the principal root.



Appendix II

Continuous Functions

The development of calculus depends in an essential way on the concept of limit of a function and thereby on properties of the real number system. In Chapter 1 we presented these notions in an intuitive way and did not attempt to prove them except in Section 1.5, where the *formal* definition of limit was given and used to verify some elementary limits and prove some simple properties of limits.

Many of the results on limits and continuity of functions stated in Chapter 1 may seem quite obvious; most students and users of calculus are not bothered by applying them without proof. Nevertheless, mathematics is a highly logical and rigorous discipline, and any statement, however obvious, that cannot be proved by strictly logical arguments from acceptable assumptions must be considered suspect. In this appendix we build upon the formal definition of limit given in Section 1.5, and combine it with the notion of *completeness* of the real number system first encountered in Section P.1 to give formal proofs of the very important results about continuous functions stated in Theorems 8 and 9 of Section 1.4, the Max-Min Theorem and the Intermediate-Value Theorem. Most of our development of calculus in this book depends essentially on these two theorems.

The branch of mathematics that deals with proofs such as these is called mathematical analysis. This subject is usually not pursued by students in introductory calculus courses but is postponed to higher years and studied by students in majors or honours programs in mathematics. It is hoped that some of this material will be of value to honours-level calculus courses and individual students with a deeper interest in understanding calculus.

Limits of Functions

At the heart of mathematical analysis is the formal definition of limit, Definition 9 in Section 1.5, which we restate as follows:

The formal definition of limit

We say that $\lim_{x \to a} f(x) = L$ if for every positive number ϵ there exists a positive number δ, depending on ϵ (i.e., $\delta = \delta(\epsilon)$), such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon.$$

Section 1.5 was marked "optional" because understanding the material presented there was not essential for learning calculus. However, that material is an *essential* prerequisite for this appendix. It is highly recommended that you go back to Section 1.5 and read it carefully, paying special attention to Examples 2 and 4, and attempt at least Exercises 31–36. These exercises provide proofs for the standard laws of limits stated in Section 1.2.

Continuous Functions

Consider the following definitions of continuity, which are equivalent to those given in Section 1.4.

DEFINITION 1

Continuity of a function at a point

A function f, defined on an open interval containing the point a, is said to be continuous at the point a if

$$\lim_{x \to a} f(x) = f(a),$$

that is, if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

DEFINITION 2

Continuity of a function on an interval

A function f is continuous on an interval if it is continuous at every point of that interval. In the case of an endpoint of a closed interval, f need only be continuous on one side. Thus, f is continuous on the interval $[a, b]$ if

$$\lim_{t \to x} f(t) = f(x)$$

for each x satisfying $a < x < b$, and

$$\lim_{t \to a+} f(t) = f(a) \qquad \text{and} \qquad \lim_{t \to b-} f(t) = f(b).$$

These concepts are illustrated in Figure II.1.

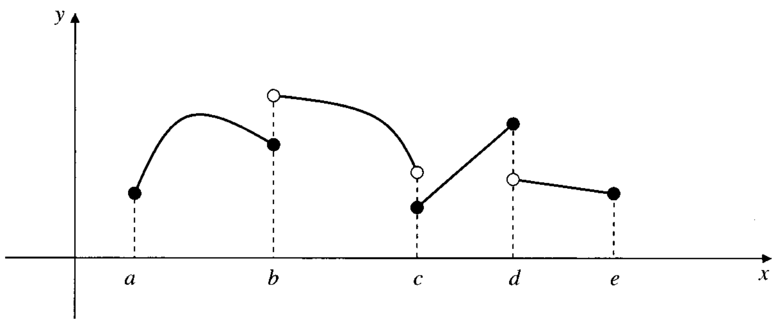


Figure II.1 f is continuous on the intervals $[a, b]$, $]b, c[$, $[c, d]$, and $]d, e]$

Some important results about continuous functions are collected in Theorems 6 and 7 of Section 1.4, which we restate here:

THEOREM 1

Combining continuous functions

(a) If f and g are continuous at the point a, then so are $f + g$, $f - g$, fg, and, if $g(a) \neq 0$, f/g.

(b) If f is continuous at the point L and if $\lim_{x \to a} g(x) = L$, then we have

$$\lim_{x \to a} f\big(g(x)\big) = f(L) = f\big(\lim_{x \to a} g(x)\big).$$

In particular, if g is continuous at the point a (so that $L = g(a)$), then $\lim_{x \to a} f\big(g(x)\big) = f\big(g(a)\big)$, that is, $f \circ g(x) = f\big(g(x)\big)$ is continuous at $x = a$.

(c) The functions $f(x) = C$ (constant) and $g(x) = x$ are continuous on the whole example,

$$\lim_{x \to a} f(x)g(x) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x)) = f(a)g(a).$$

Part (b) can be proved as follows. Let $\epsilon > 0$ be given. Since f is continuous at L, there exists $k > 0$ such that $|f\big(g(x)\big) - f(L)| < \epsilon$ whenever $|g(x) - L| < k$. Since $\lim_{x \to a} g(x) = L$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x) - L| < k$. Hence, if $0 < |x - a| < \delta$, then $|f\big(g(x)\big) - f(L)| < \epsilon$, and $\lim_{x \to a} f\big(g(x)\big) = f(L)$.

The proofs of (c) and (d) are left to the student in Exercises 3–9 at the end of this appendix.

Completeness and Sequential Limits

DEFINITION 3

A real number u is said to be an **upper bound** for a nonempty set S of real numbers if $x \le u$ for every x in S.
The number u^* is called the **least upper bound** of S if u^* is an upper bound for S and $u^* \le u$ for every upper bound u of S.
Similarly, ℓ is a **lower bound** for S if $\ell \le x$ for every x in S. The number ℓ^* is the **greatest lower bound** of S if ℓ^* is a lower bound and $\ell \le \ell^*$ for every lower bound ℓ of S.

Example 1 Set $S_1 = [2, 3]$ and $S_2 =]2, \infty[$. Any number $u \ge 3$ is an upper bound for S_1. S_2 has no upper bound; we say that it is not bounded above. The least upper bound of S_1 is 3. Any real number $\ell \le 2$ is a lower bound for both S_1 and S_2. $\ell^* = 2$ is the greatest lower bound for each set. Note that the least upper bound and greatest lower bound of a set may or may not belong to that set. ■

We now recall the completeness axiom for the real number system, which we discussed briefly in Section P.1.

The completeness axiom for the real numbers

A nonempty set of real numbers that has an upper bound must have a least upper bound.
Equivalently, a nonempty set of real numbers having a lower bound must have a greatest lower bound.

We stress that this is an *axiom* to be assumed without proof. It cannot be deduced from the more elementary algebraic and order properties of the real numbers. These other properties are shared by the rational numbers, a set that is not complete. The completeness axiom is essential for the proof of the most important results about continuous functions, in particular, for the Max-Min Theorem and the Intermediate-Value Theorem. Before attempting these proofs, however, we must develop a little more machinery.

In Section 9.1 we stated a version of the completeness axiom that pertains to *sequences* of real numbers; specifically, that an increasing sequence that is bounded above converges to a limit. We begin by verifying that this follows from the version stated above. (Both statements are, in fact, equivalent.) As noted in Section 9.1, the sequence

$$\{x_n\} = \{x_1, x_2, x_3, \ldots\}$$

is a function on the positive integers, that is, $x_n = x(n)$. We say that the sequence converges to the limit L, and we write $\lim x_n = L$, if the corresponding function $x(t)$ satisfies $\lim_{t\to\infty} x(t) = L$ as defined above. More formally,

DEFINITION 4

Limit of a sequence

We say that $\lim x_n = L$ if for every positive number ϵ there exists a positive number $N = N(\epsilon)$ such that $|x_n - L| < \epsilon$ holds whenever $n \geq N$.

THEOREM 2

If $\{x_n\}$ is an increasing sequence that is bounded above, that is,

$$x_{n+1} \geq x_n \quad \text{and} \quad x_n \leq K \quad \text{for } n = 1, 2, 3, \ldots,$$

then $\lim x_n = L$ exists. (Equivalently, if $\{x_n\}$ is decreasing and bounded below, then $\lim x_n$ exists.)

PROOF Let $\{x_n\}$ be increasing and bounded above. The set S of real numbers x_n has an upper bound, K, and so has a least upper bound, say L. Thus $x_n \leq L$ for every n, and if $\epsilon > 0$, then there exists a positive integer N such that $x_N > L - \epsilon$. (Otherwise, $L - \epsilon$ would be an upper bound for S that is lower than the least upper bound.) If $n \geq N$, then we have $L - \epsilon < x_N \leq x_n \leq L$, so $|x_n - L| < \epsilon$. Thus $\lim x_n = L$. The proof for a decreasing sequence that is bounded below is similar.

THEOREM 3

If $a \leq x_n \leq b$ for each n, and if $\lim x_n = L$, then $a \leq L \leq b$.

PROOF Suppose that $L > b$. Let $\epsilon = L - b$. Since $\lim x_n = L$, there exists n such that $|x_n - L| < \epsilon$. Thus $x_n > L - \epsilon = L - (L - b) = b$, which is a contradiction since we are given that $x_n \leq b$. Thus $L \leq b$. A similar argument shows that $L \geq a$.

THEOREM 4

If f is continuous on $[a, b]$, if $a \leq x_n \leq b$ for each n, and if $\lim x_n = L$, then $\lim f(x_n) = f(L)$.

The proof is similar to that of Theorem 1(b), and is left as Exercise 15 at the end of this appendix.

Continuous Functions on a Closed, Finite Interval

We are now in a position to prove the main results about continuous functions on closed, finite intervals.

THEOREM 5

The Boundedness Theorem

If f is continuous on $[a, b]$, then f is bounded there; that is, there exists a constant K such that $|f(x)| \leq K$ if $a \leq x \leq b$.

PROOF We show that f is bounded above; a similar proof shows that f is bounded below. For each positive integer n let S_n be the set of points x in $[a, b]$ such that $f(x) > n$:

$$S_n = \{x : a \leq x \leq b \quad \text{and} \quad f(x) > n\}.$$

We would like to show that S_n is empty for some n. It would then follow that $f(x) \leq n$ for all x in $[a, b]$; that is, n would be an upper bound for f on $[a, b]$.

Suppose, to the contrary, that S_n is nonempty for every n. We will show that this leads to a contradiction. Since S_n is bounded below (a is a lower bound), by completeness S_n has a greatest lower bound; call it x_n. (See Figure II.2.) Evidently $a \leq x_n$. Since $f(x) > n$ at some point of $[a, b]$ and f is continuous at that point, $f(x) > n$ on some interval contained in $[a, b]$. Hence $x_n < b$. It follows that $f(x_n) \geq n$. (If $f(x_n) < n$, then by continuity $f(x) < n$ for some distance to the right of x_n, and x_n could not be the greatest lower bound of S_n.)

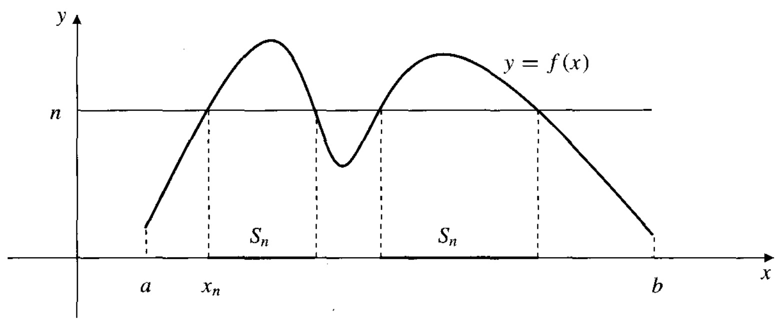


Figure II.2 The set S_n

For each n we have $S_{n+1} \subset S_n$. Therefore, $x_{n+1} \geq x_n$ and $\{x_n\}$ is an increasing sequence. Being bounded above (b is an upper bound) this sequence converges, by Theorem 2. Let $\lim x_n = L$. By Theorem 3, $a \leq L \leq b$. Since f is continuous at L, $\lim f(x_n) = f(L)$ exists by Theorem 4. But since $f(x_n) \geq n$, $\lim f(x_n)$ cannot exist. This contradiction completes the proof.

THEOREM 6

The Max-Min Theorem

If f is continuous on $[a, b]$, then there are points v and u in $[a, b]$ such that for any x in $[a, b]$ we have

$$f(v) \leq f(x) \leq f(u);$$

that is, f assumes maximum and minimum values on $[a, b]$.

PROOF By Theorem 5 we know that the set $S = \{f(x) : a \leq x \leq b\}$ has an upper bound and, therefore, by the completeness axiom, a least upper bound. Call this least upper bound M. Suppose that there exists no point u in $[a, b]$ such that $f(u) = M$. Then by Theorem 1(a), $1/(M - f(x))$ is continuous on $[a, b]$. By Theorem 5, there exists a constant K such that $1/(M - f(x)) \leq K$ for all x in $[a, b]$. Thus $f(x) \leq M - 1/K$, which contradicts the fact that M is the *least* upper bound for the values of f. Hence, there must exist some point u in $[a, b]$ such that $f(u) = M$. Since M is an upper bound for the values of f on $[a, b]$, we have $f(x) \leq f(u) = M$ for all x in $[a, b]$.

The proof that there must exist a point v in $[a, b]$ such that $f(x) \geq f(v)$ for all x in $[a, b]$ is similar.

THEOREM 7

The Intermediate-Value Theorem

If f is continuous on $[a, b]$ and s is a real number lying between the numbers $f(a)$ and $f(b)$, then there exists a point c in $[a, b]$ such that $f(c) = s$.

PROOF To be specific, we assume that $f(a) < s < f(b)$. (The proof for the case $f(a) > s > f(b)$ is similar.) Let $S = \{x : a \leq x \leq b \text{ and } f(x) \leq s\}$. S is nonempty (a belongs to S) and bounded above (b is an upper bound), so by completeness S has a least upper bound; call it c.

Suppose that $f(c) > s$. Then $c \neq a$ and, by continuity, $f(x) > s$ on some interval $]c - \delta, c]$ where $\delta > 0$. But this says $c - \delta$ is an upper bound for S lower than the least upper bound, which is impossible. Thus $f(c) \leq s$.

Suppose $f(c) < s$. Then $c \neq b$ and, by continuity, $f(x) < s$ on some interval of the form $[c, c + \delta[$ for some $\delta > 0$. But this says that $[c, c + \delta[\subset S$, which contradicts the fact that c is an upper bound for S. Hence we cannot have $f(c) < s$. Therefore, $f(c) = s$.

For more discussion of these theorems, and some applications, see Section 1.4.

Exercises

1. Let $a < b < c$ and suppose that $f(x) \leq g(x)$ for $a \leq x \leq c$. If $\lim_{x\to b} f(x) = L$ and $\lim_{x\to b} g(x) = M$, prove that $L \leq M$. *Hint:* assume that $L > M$ and deduce that $f(x) > g(x)$ for all x sufficiently near b. This contradicts the condition that $f(x) \leq g(x)$ for $a \leq x \leq b$.
2. If $f(x) \leq K$ on the intervals $[a, b)$ and $(b, c]$, and if $\lim_{x\to b} f(x) = L$, prove that $L \leq K$.
3. Use the formal definition of limit to prove that $\lim_{x\to 0+} x^r = 0$ for any positive, rational number r.

Prove the assertions in Exercises 4–9.

4. $f(x) = C$ (constant) and $g(x) = x$ are both continuous on the whole real line.
5. Every polynomial is continuous on the whole real line.
6. A rational function (quotient of polynomials) is continuous everywhere except where the denominator is 0.
7. If n is a positive integer and $a > 0$, then $f(x) = x^{1/n}$ is continuous at $x = a$.
8. If $r = m/n$ is a rational number, then $g(x) = x^r$ is continuous at every point $a > 0$.
9. If $r = m/n$, where m and n are integers and n is odd, show that $g(x) = x^r$ is continuous at every point $a < 0$. If $r \geq 0$, show that g is continuous at 0 also.
10. Prove that $f(x) = |x|$ is continuous on the real line.

Use the definitions from Chapter 3 for the functions in Exercises 11–14 to show that these functions are continuous on their respective domains.

11. $\sin x$

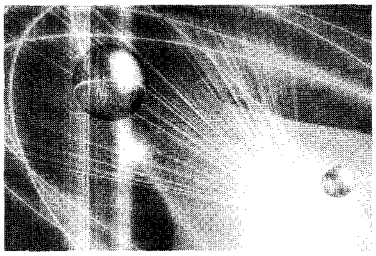
12. $\cos x$

13. $\ln x$

14. e^x

15. Prove Theorem 4.

16. Suppose that every function that is continuous and bounded on $[a, b]$ must assume a maximum value and a minimum value on that interval. Without using Theorem 5, prove that every function f that is continuous on $[a, b]$ must be bounded on that interval. *Hint:* show that $g(t) = t/(1 + |t|)$ is continuous and increasing on the real line. Then consider $g\big(f(x)\big)$.



Appendix III

The Riemann Integral

In Section 5.3 we defined the definite integral $\int_a^b f(x)\,dx$ of a function f that is continuous on the finite, closed interval $[a, b]$. The integral was defined as a kind of "limit" of Riemann sums formed by partitioning the interval $[a, b]$ into small subintervals. In this appendix we will reformulate the definition of the integral so that it can be used for functions that are not necessarily continuous; in the following discussion we assume only that f is **bounded** on $[a, b]$. Later we will prove Theorem 2 of Section 5.3, which asserts that any continuous function is integrable.

Recall that a **partition** P of $[a, b]$ is a finite, ordered set of points $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Such a partition subdivides $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$, where $n = n(P)$ depends on the partition. The length of the jth subinterval $[x_{j-1}, x_j]$ is $\Delta x_j = x_j - x_{j-1}$.

Suppose that the function f is bounded on $[a, b]$. Given any partition P, the n sets $S_j = \{f(x) : x_{j-1} \le x \le x_j\}$ have least upper bounds M_j and greatest lower bounds m_j, $(1 \le j \le n)$, so that

$$m_j \le f(x) \le M_j \qquad \text{on} \qquad [x_{j-1}, x_j].$$

We define upper and lower Riemann sums for f corresponding to the partition P to be

$$U(f, P) = \sum_{j=1}^{n(P)} M_j \Delta x_j \qquad \text{and}$$

$$L(f, P) = \sum_{j=1}^{n(P)} m_j \Delta x_j.$$

(See Figure III.1.) Note that if f is continuous on $[a, b]$, then m_j and M_j are, in fact, the minimum and maximum values of f over $[x_{j-1}, x_j]$ (by Theorem 6 of Appendix II); that is, $m_j = f(l_j)$ and $M_j = f(u_j)$, where $f(l_j) \le f(x) \le f(u_j)$ for $x_{j-1} \le x \le x_j$.

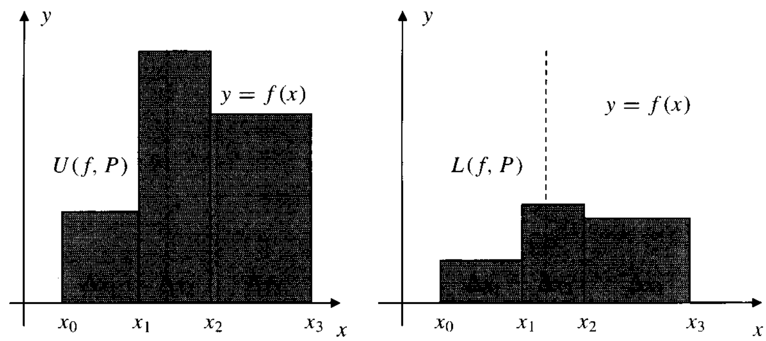


Figure III.1 Upper and lower sums corresponding to the partition $P = \{x_0, x_1, x_2, x_3\}$

If P is any partition of $[a, b]$ and we create a new partition P^* by adding new subdivision points to those of P, thus subdividing the subintervals of P into smaller ones, then we call P^* a **refinement** of P.

THEOREM 1

If P^* is a refinement of P, then $L(f, P^*) \geq L(f, P)$ and $U(f, P^*) \leq U(f, P)$.

PROOF If S and T are sets of real numbers, and $S \subset T$, then any lower bound (or upper bound) of T is also a lower bound (or upper bound) of S. Hence, the greatest lower bound of S is at least as large as that of T, and the least upper bound of S is no greater than that of T.

Let P be a given partition of $[a, b]$ and form a new partition P' by adding one subdivision point to those of P, say the point k dividing the jth subinterval $[x_{j-1}, x_j]$ of P into two subintervals $[x_{j-1}, k]$ and $[k, x_j]$. (See Figure III.2.) Let m_j, m'_j, and m''_j be the greatest lower bounds of the sets of values of $f(x)$ on the intervals $[x_{j-1}, x_j]$, $[x_{j-1}, k]$, and $[k, x_j]$, respectively. Then $m_j \leq m'_j$ and $m_j \leq m''_j$. Thus $m_j(x_j - x_{j-1}) \leq m'_j(k - x_{j-1}) + m''_j(x_j - k)$, so $L(f, P) \leq L(f, P')$.

If P^* is a refinement of P, it can be obtained by adding one point at a time to those of P and thus $L(f, P) \leq L(f, P^*)$. We can prove that $U(f, P) \geq U(f, P^*)$ in a similar manner.

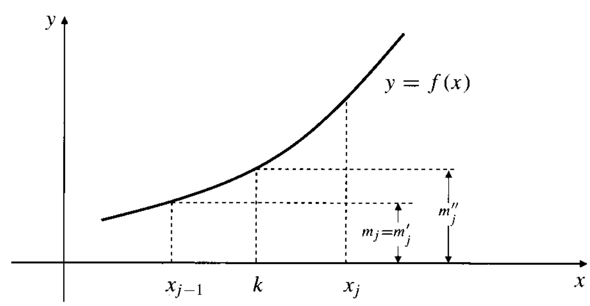


Figure III.2 Adding one point to a partition

THEOREM 2

If P and P' are any two partitions of $[a, b]$, then $L(f, P) \leq U(f, P')$.

PROOF Combine the subdivision points of P and P' to form a new partition P^*, which is a refinement of both P and P'. Then by Theorem 1,

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P').$$

No lower sum can exceed any upper sum.

Theorem 2 shows that the set of values of $L(f, P)$ for fixed f and various partitions P of $[a, b]$ is a bounded set; any upper sum is an upper bound for this set. By completeness, the set has a least upper bound, which we shall denote I_*. Thus, $L(f, P) \le I_*$ for any partition P. Similarly, there exists a greatest lower bound I^* for the set of values of $U(f, P)$ corresponding to different partitions P. It follows that $I_* \le I^*$. (See Exercise 4 at the end of this appendix.)

DEFINITION 1

The Riemann integral

If f is bounded on $[a, b]$ and $I_* = I^*$, then we say that f is **Riemann integrable**, or simply **integrable** on $[a, b]$, and denote by

$$\int_a^b f(x)\,dx = I_* = I^*$$

the **(Riemann) integral** of f on $[a, b]$.

The following theorem provides a convenient test for determining whether a given bounded function is integrable:

THEOREM 3

The bounded function f is integrable on $[a, b]$ if and only if for every positive number ϵ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

PROOF Suppose that for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$, then

$$I^* \le U(f, P) < L(f, P) + \epsilon \le I_* + \epsilon.$$

Since $I^* < I_* + \epsilon$ must hold for every $\epsilon > 0$, it follows that $I^* \le I_*$. Since we already know that $I^* \ge I_*$, we have $I^* = I_*$ and f is integrable on $[a, b]$.

Conversely, if $I^* = I_*$ and $\epsilon > 0$ are given, we can find a partition P' such that $L(f, P') > I_* - \epsilon/2$, and another partition P'' such that $U(f, P'') < I^* + \epsilon/2$. If P is a common refinement of P' and P'', then by Theorem 1 we have that $U(f, P) - L(f, P) \le U(f, P'') - L(f, P') < (\epsilon/2) + (\epsilon/2) = \epsilon$, as required.

Example 1 Let $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \text{ or } 1 < x \le 2 \\ 1 & \text{if } x = 1. \end{cases}$

Show that f is integrable on $[0, 2]$ and find $\int_0^2 f(x)\,dx$.

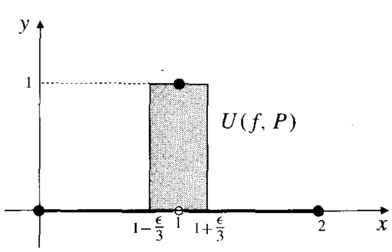


Figure III.3

Solution Let $\epsilon > 0$ be given. Let $P = \{0,\ 1-\epsilon/3,\ 1+\epsilon/3,\ 2\}$. Then $L(f, P) = 0$ since $f(x) = 0$ at points of each of these subintervals into which P subdivides $[0, 2]$. (See Figure III.3.) Since $f(1) = 1$, we have

$$U(f, P) = 0\left(1 - \frac{\epsilon}{3}\right) + 1\left(\frac{2\epsilon}{3}\right) + 0\left(2 - \left(1 + \frac{\epsilon}{3}\right)\right) = \frac{2\epsilon}{3}.$$

Hence, $U(f, P) - L(f, P) < \epsilon$ and f is integrable on $[0, 2]$. Since $L(f, P) = 0$ for every partition, $\int_0^2 f(x)\,dx = I_* = 0$.

Example 2 Let $f(x)$ be defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is not integrable on $[0, 1]$.

PROOF Every subinterval of $[0, 1]$ having positive length contains both rational and irrational numbers. Hence, for any partition P of $[0, 1]$ we have $L(f, P) = 0$ and $U(f, P) = 1$. Thus $I_* = 0$ and $I^* = 1$, so f is not integrable on $[0, 1]$. ■

Uniform Continuity

When we assert that a function f is continuous on the interval $[a, b]$, we imply that for every x in that interval and every $\epsilon > 0$, we can find a positive number δ (depending on *both* x and ϵ) such that $|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \delta$ and y lies in $[a, b]$. In fact, however, it is possible to find a number δ *depending only on* ϵ such that $|f(y) - f(x)| < \epsilon$ holds whenever x and y belong to $[a, b]$ and satisfy $|y - x| < \delta$. We describe this phenomenon by saying that f is **uniformly continuous** on the interval $[a, b]$.

THEOREM 4

If f is continuous on the closed, finite interval $[a, b]$, then f is uniformly continuous on that interval.

PROOF Let $\epsilon > 0$ be given. Define numbers x_n in $[a, b]$ and subsets S_n of $[a, b]$ as follows:

$$x_1 = a$$
$$S_1 = \left\{x : x_1 < x \le b \text{ and } |f(x) - f(x_1)| \ge \frac{\epsilon}{3}\right\}.$$

If S_1 is empty, stop; otherwise, let

$$x_2 = \text{ the greatest lower bound of } S_1$$
$$S_2 = \left\{x : x_2 < x \le b \text{ and } |f(x) - f(x_2)| \ge \frac{\epsilon}{3}\right\}.$$

If S_2 is empty, stop; otherwise, proceed to define x_3 and S_3 analogously. We proceed in this way as long as we can; if x_n and S_n have been defined and S_n is not empty, we define

$$x_{n+1} = \text{ the greatest lower bound of } S_n$$
$$S_{n+1} = \left\{x : x_{n+1} < x \le b \text{ and } |f(x) - f(x_{n+1})| \ge \frac{\epsilon}{3}\right\}.$$

At any stage where S_n is not empty, the continuity of f at x_n assures us that $x_{n+1} > x_n$ and $|f(x_{n+1}) - f(x_n)| = \epsilon/3$.

We must consider two possibilities for the above procedure: either S_n is empty for some n, or S_n is nonempty for every n.

Suppose S_n is nonempty for every n. Then we have constructed an infinite, increasing sequence $\{x_n\}$ in $[a, b]$ that, being bounded above (by b), must have a limit by completeness (Theorem 2 of Appendix II). Let $\lim x_n = x^*$. We have $a \le x^* \le b$. Since f is continuous at x^*, there exists $\delta > 0$ such that

$|f(x)-f(x^*)| < \epsilon/8$ whenever $|x-x^*| < \delta$ and x lies in $[a, b]$. Since $\lim x_n = x^*$, there exists a positive integer N such that $|x_n - x^*| < \delta$ whenever $n \geq N$. For such n we have

$$\begin{aligned}\frac{\epsilon}{3} = |f(x_{n+1}) - f(x_n)| &= |f(x_{n+1}) - f(x^*) + f(x^*) - f(x_n)| \\ &\leq |f(x_{n+1}) - f(x^*)| + |f(x_n) - f(x^*)| \\ &< \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{4},\end{aligned}$$

which is clearly impossible. Thus S_n must, in fact, be empty for some n.

Suppose that S_N is empty. Thus, S_n is nonempty for $n < N$, and the procedure for defining x_n stops with x_N. Since S_{N-1} is not empty, $x_N < b$. In this case define $x_{N+1} = b$ and let

$$\delta = \min\{x_2 - x_1,\ x_3 - x_2,\ \ldots,\ x_{N+1} - x_N\}.$$

The minimum of a finite set of positive numbers is a positive number, so $\delta > 0$. If x lies in $[a, b]$, then x lies in one of the intervals $[x_1, x_2]$, $[x_2, x_3]$, ..., $[x_N, x_{N+1}]$. Suppose x lies in $[x_k, x_{k+1}]$. If y is in $[a, b]$ and $|y - x| < \delta$, then y lies in either the same subinterval as x or in an adjacent one; that is, y lies in $[x_j, x_{j+1}]$, where $j = k - 1, k$, or $k + 1$. Thus,

$$\begin{aligned}|f(y) - f(x)| &= |f(y) - f(x_j) + f(x_j) - f(x_k) + f(x_k) - f(x)| \\ &\leq |f(y) - f(x_j)| + |f(x_j) - f(x_k)| + |f(x_k) - f(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,\end{aligned}$$

which was to be proved.

We are now in a position to prove that a continuous function is integrable.

THEOREM 5 If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

PROOF By Theorem 4, f is uniformly continuous on $[a, b]$. Let $\epsilon > 0$ be given. Let $\delta > 0$ be such that $|f(x) - f(y)| < \epsilon/(b - a)$ whenever $|x - y| < \delta$ and x and y belong to $[a, b]$. Choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ for which each subinterval $[x_{j-1}, x_j]$ has length $\Delta x_j < \delta$. Then the greatest lower bound, m_j, and the least upper bound, M_j, of the set of values of $f(x)$ on $[x_{j-1}, x_j]$ satisfy $M_j - m_j < \epsilon/(b - a)$. Accordingly,

$$U(f, P) - L(f, P) < \frac{\epsilon}{b-a} \sum_{j=1}^{n(P)} \Delta x_j = \frac{\epsilon}{b-a}(b - a) = \epsilon.$$

Thus f is integrable on $[a, b]$, as asserted.

Exercises

1. Let $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \end{cases}$. Prove that f is integrable on $[0, 2]$ and find the value of $\int_0^2 f(x)\,dx$.

2. Let $f(x) = \begin{cases} 1 & \text{if } x = 1/n, \quad n = 1, 2, 3, \dots \\ 0 & \text{for all other values of } x \end{cases}$.
Show that f is integrable over $[0, 1]$ and find the value of the integral $\int_0^1 f(x)\,dx$.

* 3. Let $f(x) = 1/n$ if $x = m/n$, where m, n are integers having no common factors, and let $f(x) = 0$ if x is an irrational number. Thus, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, etc. Show that f is integrable on $[0, 1]$ and find $\int_0^1 f(x)\,dx$. *Hint:* show that for any $\epsilon > 0$, only finitely many points of the graph of f over $[0, 1]$ lie above the line $y = \epsilon$.

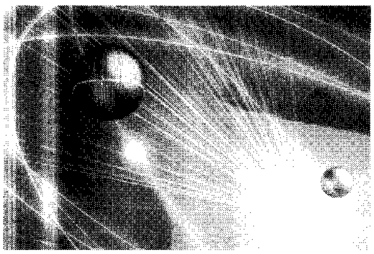
4. Prove that I_* and I^* defined in the paragraph following Theorem 2 satisfy $I_* \le I^*$ as claimed there.

5. Prove parts (c), (d), (e), (f), (g), and (h) of Theorem 3 in Section 5.4 for the Riemann integral.

6. Use the definition of uniform continuity given in the paragraph preceding Theorem 4 to prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$. Do not use Theorem Theorem 4 itself.

7. Show directly from the definition of uniform continuity (without using Theorem 5 of Appendix II) that a function f uniformly continuous on a closed, finite interval is necessarily bounded there.

8. If f is bounded and integrable on $[a, b]$, prove that $F(x) = \int_a^x f(t)\,dt$ is uniformly continuous on $[a, b]$. (If f were continuous, we would have a stronger result; F would be differentiable on (a, b) and $F'(x) = f(x)$ (which is the Fundamental Theorem of Calculus).)



Appendix IV

Differential Equations

Introduction A **differential equation** (or **DE**) is an equation that involves one or more derivatives of an unknown function. Solving the differential equation means finding a function (or every such function) that satisfies the differential equation.

Many physical laws and relationships between quantities studied in various scientific disciplines are expressed mathematically as differential equations. For example, Newton's Second Law of Motion ($F = ma$) states that the position $x(t)$ at time t of an object of constant mass m subjected to a force $F(t)$ must satisfy the differential equation (equation of motion):

$$m\frac{d^2x}{dt^2} = F(t).$$

Similarly, the biomass $m(t)$ at time t of a bacterial culture growing in a uniformly supporting medium changes at a rate proportional to the biomass:

$$\frac{dm}{dt} = km(t),$$

which is the differential equation of exponential growth (or, if $k < 0$, exponential decay). Because differential equations arise so extensively in the abstract modelling of concrete phenomena, such equations and techniques for solving them are at the heart of applied mathematics. Indeed, most of the existing mathematical literature is either directly involved with differential equations or is motivated by problems arising in the study of such equations. Because of this, various differential equations, terms for their description, and techniques for their solution are introduced throughout *Calculus: A Complete Course.* This appendix provides some introductory background not covered elsewhere in the book. However, students of mathematics and its applications usually take one or more full courses on differential equations, and even then hardly scratch the surface of the subject.

Classifying Differential Equations

Differential equations are classified in several ways. The most significant classification is based on the number of variables with respect to which derivatives appear

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

models the lateral displacement $u(x, t)$ at position x at time t of a stretched vibrating string. We will not discuss partial differential equations in this appendix.

Differential equations are also classified with respect to **order**. The order of a differential equation is the order of the highest-order derivative present in the equation. The one-dimensional wave equation is a second-order PDE. The following example records the order of two ODEs.

Example 1

$$\frac{d^2 y}{dx^2} + x^3 y = \sin x \qquad \text{has order 2,}$$

$$\frac{d^3 y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 = y\,\frac{d^2 y}{dx^2} + e^y \qquad \text{has order 3.}$$

Like any equation, a differential equation can be written in the form $F = 0$, where F is a function. For an ODE, the function F can depend on the independent variable (usually called x or t), the unknown function (usually y), and any derivatives of the unknown function up to the order of the equation. For instance, an nth-order ODE can be written in the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

Linear ODEs

An important special class of differential equations consists of those that are **linear**. An nth-order linear ODE has the form

$$\begin{aligned} a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots \\ + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x), \end{aligned}$$

or, more simply,

$$P_n(D)y(x) = f(x),$$

where $P_n(D)$ is the nth-order differential operator

$$P_n(D) = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_2(x)D^2 + a_1(x)D + a_0(x)$$

obtained by substituting the differential operator $D = d/dx$ for the variable r in the nth-degree polynomial

$$P_n(r) = a_n(x)r^n + a_{n-1}(x)r^{n-1} + \cdots + a_2(x)r^2 + a_1(x)r + a_0(x),$$

having coefficients depending on the variable x. It is often useful to write linear DEs in terms of differential operators in this way.

Each term in the expression on the left side of the linear DE is the product of a *coefficient* that is a function of x and a second factor that is either y or one of the derivatives of y. The term $f(x)$ on the right does not depend on y; it is called the **nonhomogeneous term**.

A linear ODE is said to be **homogeneous** if all of its terms involve the unknown function y, that is, if $f(x)$ is identically zero. If $f(x)$ is not identically zero, the equation is **nonhomogeneous**.

Example 2 The first DE in Example 1,

$$\frac{d^2y}{dx^2} + x^3y = \sin x,$$

is linear and nonhomogeneous. Here, the coefficients are $a_2(x) = 1$, $a_1(x) = 0$, and $a_0(x) = x^3$, and the nonhomogeneous term is $f(x) = \sin x$. Although it can be written in the form

$$\frac{d^3y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} - e^y = 0,$$

the second equation in Example 1 is *not linear* (we say it is **nonlinear**) because the second term involves the square of a derivative of y, the third term involves the product of y and one of its derivatives, and the fourth term is not y times a function of x. The equation

$$(1 + x^2)\frac{d^3y}{dx^3} + \sin x\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

is a linear equation of order 3. The coefficients are $a_3(x) = 1 + x^2$, $a_2(x) = \sin x$, $a_1(x) = -4$, and $a_0(x) = 1$. Since $f(x) = 0$, this equation is *homogeneous*. ■

The following theorem states that any *linear combination* of solutions of a linear, homogeneous DE is also a solution. This is an extremely important fact about linear, homogeneous DEs.

THEOREM 1

If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of the linear, homogeneous DE

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = 0,$$

then so is the linear combination

$$y = Ay_1(x) + By_2(x)$$

for any values of the constants A and B.

PROOF We are given that

$$a_n y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \cdots + a_2 y_1'' + a_1 y_1' + a_0 y_1 = 0 \qquad \text{and}$$
$$a_n y_2^{(n)} + a_{n-1} y_2^{(n-1)} + \cdots + a_2 y_2'' + a_1 y_2' + a_0 y_2 = 0.$$

Multiplying the first equation by A and the second by B and adding the two gives

$$a_n (Ay_1^{(n)} + By_2^{(n)}) + a_{n-1}(Ay_1^{(n-1)} + By_2^{(n-1)}) + \cdots + a_2(Ay_1'' + By_2'') + a_1(Ay_1' + By_2') + a_0(Ay_1 + By_2) = 0.$$

Thus, $y = Ay_1(x) + By_2(x)$ is also a solution of the equation.

The same kind of proof can be used to verify the following theorem.

THEOREM 2 If $y = y_1(x)$ is a solution of the linear, homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

and $y = y_2(x)$ is a solution of the linear, nonhomogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x),$$

then $y = y_1(x) + y_2(x)$ is also a solution of the same linear, nonhomogeneous equation.

We made extensive use of these two facts when we discussed second-order linear equations with constant coefficients in Section 3.7.

First-Order ODEs

We have discussed techniques for solving several kinds of first-order DEs in various sections of this book:

- Equations of the form $\dfrac{dy}{dx} = f(x)$ were discussed in Section 2.10.
- Equations of the form $\dfrac{dy}{dx} = f(x)g(y)$ (called **separable equations**) were discussed in Section 7.9
- Equations of the form $\dfrac{dy}{dx} + p(x)y = q(x)$ (which are **linear** and **nonhomogeneous**) were also treated in Section 7.9.

Unfortunately, the term *homogeneous* is used in more than one way in the study of differential equations. Certain first-order ODEs that are not necessarily linear are called homogeneous for a different reason than the one applying for linear equations above. A first-order DE of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

is said to be **homogeneous** because y/x and, therefore, $g(x, y) = f(y/x)$ are *homogeneous of degree 0* in the sense described in Section 12.5. Such a homogeneous equation can be transformed into a separable equation (and therefore solved) by means of a change of dependent variable. If we set

$$v = \frac{y}{x}, \qquad \text{or, equivalently,} \qquad y = xv(x),$$

then we have

$$\frac{dy}{dx} = v + x\,\frac{dv}{dx},$$

and the original differential equation transforms into

$$\frac{dv}{dx} = \frac{f(v) - v}{x},$$

which is separable.

Example 3 Solve the equation

$$\frac{dy}{dx} = \frac{x^2 + xy}{xy + y^2}.$$

Solution The equation is homogeneous. (Divide the numerator and denominator of the right-hand side by x^2 to see this.) If $y = vx$, the equation becomes

$$v + x\,\frac{dv}{dx} = \frac{1+v}{v+v^2} = \frac{1}{v},$$

or

$$x\,\frac{dv}{dx} = \frac{1 - v^2}{v}.$$

Separating variables and integrating, we calculate

$$\begin{aligned}
\int \frac{v\,dv}{1-v^2} &= \int \frac{dx}{x} && \text{Let } u = 1 - v^2. \\
-\frac{1}{2}\int \frac{du}{u} &= \int \frac{dx}{x} \\
-\ln|u| &= 2\ln|x| + C_1 = \ln C_2 x^2 && (C_1 = \ln C_2). \\
\frac{1}{|u|} &= C_2 x^2 \\
|1 - v^2| &= \frac{C_3}{x^2} && (C_3 = 1/C_2). \\
\left|1 - \frac{y^2}{x^2}\right| &= \frac{C_3}{x^2}.
\end{aligned}$$

The solution is best expressed in the form $x^2 - y^2 = C_4$. However, near points where $y \neq 0$, the equation can be solved for y as a function of x. ■

Exact Equations

A first-order differential equation expressed in differential form as

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

which is equivalent to $\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$, is said to be **exact** if the left-hand side is the differential of a function $\phi(x, y)$:

$$d\phi(x, y) = M(x, y)\,dx + N(x, y)\,dy.$$

The function ϕ is called an **integral function** of the differential equation. The level curves $\phi(x, y) = C$ of ϕ are the **solution curves** of the differential equation. For example, the differential equation

$$x\,dx + y\,dy = 0$$

has solution curves given by

$$x^2 + y^2 = C$$

since $d(x^2 + y^2) = 2(x\,dx + y\,dy) = 0$.

Remark The condition that the differential equation $M\,dx + N\,dy = 0$ should be exact is just the condition that the vector field

$$\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$$

should be *conservative*; the integral function of the differential equation is then the potential function of the vector field. (See Section 15.2.)

A **necessary condition** for the exactness of the DE $M\,dx + N\,dy = 0$ is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x};$$

this just says that the mixed partial derivatives $\frac{\partial^2 \phi}{\partial x \partial y}$ and $\frac{\partial^2 \phi}{\partial y \partial x}$ of the integral function ϕ must be equal.

Once you know that an equation is exact, you can often guess the integral function. In any event, ϕ can always be found by the same method used to find the potential of a conservative vector field in Section 15.2.

Example 4 Verify that the DE

$$(2x + \sin y - ye^{-x})\,dx + (x \cos y + \cos y + e^{-x})\,dy = 0$$

is exact and find its solution curves.

Solution Here, $M = 2x + \sin y - ye^{-x}$ and $N = x \cos y + \cos y + e^{-x}$. Since

$$\frac{\partial M}{\partial y} = \cos y - e^{-x} = \frac{\partial N}{\partial x},$$

the DE is exact. We want to find ϕ so that

$$\frac{\partial \phi}{\partial x} = M = 2x + \sin y - ye^{-x} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N = x \cos y + \cos y + e^{-x}.$$

Integrate the first equation with respect to x, being careful to allow the constant of integration to depend on y:

$$\phi(x, y) = \int (2x + \sin y - ye^{-x})\, dx = x^2 + x \sin y + ye^{-x} + C_1(y).$$

Now substitute this expression into the second equation:

$$x \cos y + \cos y + e^{-x} = \frac{\partial \phi}{\partial y} = x \cos y + e^{-x} + C_1'(y).$$

Thus $C_1'(y) = \cos y$, and $C_1(y) = \sin y + C_2$. (It is because the original DE was exact that the equation for $C_1'(y)$ turned out to be independent of x; this had to happen or we could not have found C_1 as a function of y only.) Choosing $C_2 = 0$, we find that $\phi(x, y) = x^2 + x \sin y + ye^{-x} + \sin y$ is an integral function for the given DE. The solution curves for the DE are the level curves

$$x^2 + x \sin y + ye^{-x} + \sin y = C.$$

Integrating Factors

Any ordinary differential equation of order 1 and degree 1 can be expressed in differential form: $M\, dx + N\, dy = 0$. However, this latter equation will usually not be exact. It *may* be possible to multiply the equation by an **integrating factor** $\mu(x, y)$ so that the resulting equation

$$\mu(x, y)\, M(x, y)\, dx + \mu(x, y)\, N(x, y)\, dy = 0$$

is exact. In general, such integrating factors are difficult to find; they must satisfy the partial differential equation

$$M(x, y) \frac{\partial \mu}{\partial y} - N(x, y) \frac{\partial \mu}{\partial x} = \mu(x, y) \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right),$$

which follows from the necessary condition for exactness stated above. We will not try to solve this equation here.

Sometimes it happens that a differential equation has an integrating factor depending on only one of the two variables. Suppose, for instance, that $\mu(x)$ is an integrating factor for $M\, dx + N\, dy = 0$. Then $\mu(x)$ must satisfy the ordinary differential equation

$$N(x, y) \frac{d\mu}{dx} = \mu(x) \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right),$$

or

$$\frac{1}{\mu(x)} \frac{d\mu}{dx} = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N(x, y)}.$$

This equation can be solved (by integration) for μ as a function of x alone *provided that the right-hand side is independent of* y.

Example 5 Show that $(x + y^2)\,dx + xy\,dy = 0$ has an integrating factor depending only on x, find it, and solve the equation.

Solution Here $M = x + y^2$ and $N = xy$. Since

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N(x,y)} = \frac{2y - y}{xy} = \frac{1}{x}$$

does not depend on y, the equation has an integrating factor depending only on x. This factor is given by $d\mu/\mu = dx/x$. Evidently $\mu = x$ is a suitable integrating factor; if we multiply the given differential equation by x, we obtain

$$0 = (x^2 + xy^2)\,dx + x^2y\,dy = d\left(\frac{x^3}{3} + \frac{x^2y^2}{2}\right).$$

The solution is therefore $2x^3 + 3x^2y^2 = C$.

Remark Of course, it may be possible to find an integrating factor depending on y instead of x. See Exercises 34–36 below. It is also possible to look for integrating factors that depend on specific combinations of x and y, for instance, xy. See Exercise 37.

Slope Fields and Solution Curves

A general first-order differential equation of the form

$$\frac{dy}{dx} = f(x,y)$$

specifies a slope $f(x,y)$ at every point (x,y) in the domain of f and therefore represents a **slope field**. Such a slope field can be represented graphically by drawing short line segments of the indicated slope at many points in the xy-plane. Slope fields resemble vector fields, but the segments are usually drawn having the same length and without arrowheads. Figure IV.1 portrays the slope field for the differential equation

$$\frac{dy}{dx} = x - y.$$

Solving a typical initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

involves finding a function $y = \phi(x)$ such that

$$\phi'(x) = f\big(x, \phi(x)\big) \qquad \text{and} \qquad \phi(x_0) = y_0.$$

The graph of the equation $y = \phi(x)$ is a curve passing through (x_0, y_0) that is tangent to the slope-field at each point. Such curves are called **solution curves** of the differential equation. Figure IV.1 shows four solution curves for $y' = x - y$ corresponding to the initial conditions $y(0) = C$, where $C = -2, -1, 0$, and 1.

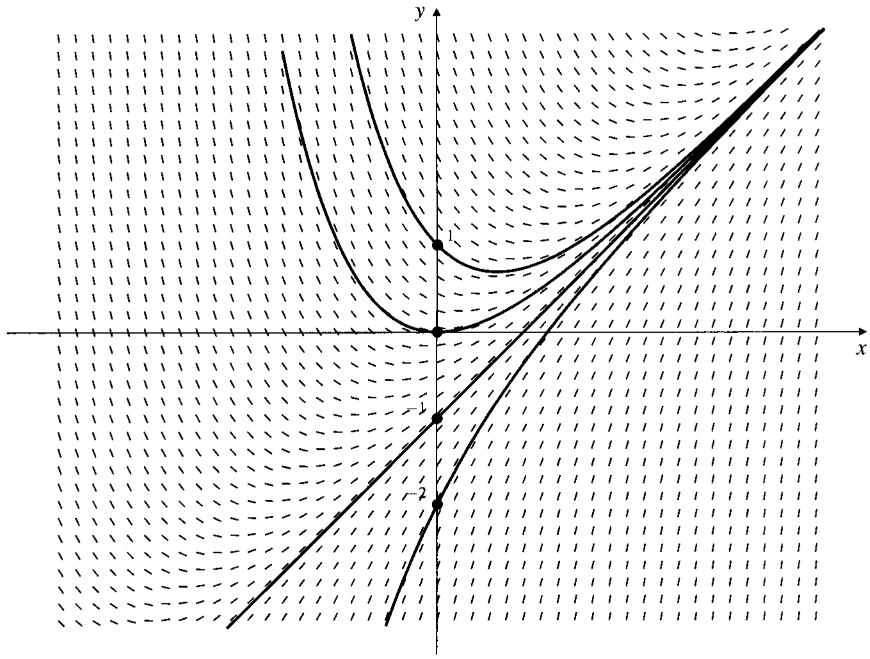


Figure IV.1 The slope field for the DE $y' = x - y$ and four solution curves for this DE

The DE $y' = x - y$ is linear and can be solved explicitly by the method of Section 7.9. Indeed, the solution satisfying $y(0) = C$ is $y = x - 1 + (C + 1)e^{-x}$. Most differential equations of the form $y' = f(x, y)$ cannot be solved for y as an explicit function of x, so we must use numerical approximation methods to find the value of a solution function $\phi(x)$ at particular points.

Existence and Uniqueness of Solutions

Even if we cannot calculate an explicit solution of an initial-value problem, it is important to know when the problem has a solution and whether that solution is unique.

THEOREM 3

An existence and uniqueness theorem for first-order initial-value problems

Suppose that $f(x, y)$ and $f_2(x, y) = (\partial/\partial y)f(x, y)$ are continuous on a rectangle R of the form $a \le x \le b$, $c \le y \le d$, containing the point (x_0, y_0) in its interior. Then there exists a number $\delta > 0$ and a *unique* function $\phi(x)$ defined and having a continuous derivative on the interval $(x_0 - \delta, x_0 + \delta)$ such that $\phi(x_0) = y_0$ and $\phi'(x) = f\big(x, \phi(x)\big)$ for $x_0 - \delta < x < x_0 + \delta$. In other words, the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases} \qquad (*)$$

has a unique solution on $(x_0 - \delta, x_0 + \delta)$.

We give only an outline of the proof here. Any solution $y = \phi(x)$ of the initial-value problem $(*)$ must also satisfy the **integral equation**

$$\phi(x) = y_0 + \int_{x_0}^{x} f\big(t, \phi(t)\big)\,dt, \qquad (**)$$

and, conversely, any solution of the integral equation (**) must also satisfy the initial-value problem (*). A sequence of approximations $\phi_n(x)$ to a solution of (**) can be constructed as follows:

$$\phi_0(x) = y_0$$
$$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f\big(t, \phi_n(t)\big)\,dt \qquad \text{for} \quad n = 0, 1, 2, \ldots.$$

(These are called **Picard iterations.**) The proof of Theorem 3 involves showing that

$$\lim_{n\to\infty} \phi_n(x) = \phi(x)$$

exists on an interval $(x_0 - \delta, x_0 + \delta)$ and that the resulting limit $\phi(x)$ satisfies the integral equation (**). The details can be found in more advanced texts on differential equations and analysis.

Remark Some initial-value problems can have nonunique solutions. For example, the functions $y_1(x) = x^3$ and $y_2(x) = 0$ both satisfy the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = 3y^{2/3} \\ y(0) = 0. \end{cases}$$

In this case $f(x, y) = 3y^{2/3}$ is continuous on the whole xy-plane. However, $\partial f/\partial y = 2y^{-1/3}$ is not continuous on the x-axis and is therefore not continuous on any rectangle containing $(0, 0)$ in its interior. The conditions of Theorem 3 are not satisfied and the initial-value problem has a solution but not a unique one.

Remark The unique solution $y = \phi(x)$ to the initial-value problem (*) guaranteed by Theorem 3 may not be defined on the whole interval $[a, b]$, because it can "escape" from the rectangle R through the top or bottom edges. Even if $f(x, y)$ and $(\partial/\partial y) f(x, y)$ are continuous on the whole xy-plane, the solution may not be defined on the whole real line. For example

$$y = \frac{1}{1 - x} \quad \text{satisfies the initial-value problem} \quad \begin{cases} \dfrac{dy}{dx} = y^2 \\ y(0) = 1 \end{cases}$$

but only for $x < 1$. Starting from $(0, 1)$, we can follow the solution curve as far as we want to the left of $x = 0$, but to the right of $x = 0$ the curve recedes to ∞ as $x \to 1-$. It makes no sense to regard the part of the curve to the right of $x = 1$ as part of the solution curve to the initial-value problem. (See Figure IV.2.)

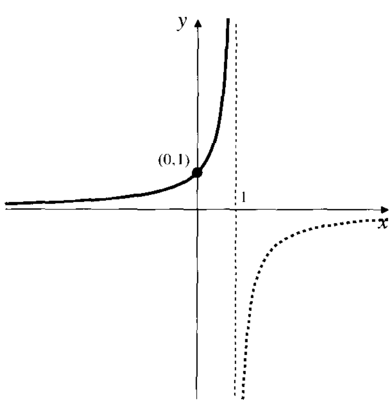


Figure IV.2 The solution to $y' = y^2$, $y(0) = 1$ is the part of the curve $y = 1/(1 - x)$ to the left of the vertical asymptote at $x = 1$

Numerical Methods

Suppose that the conditions of Theorem 3 are satisfied, so we know that the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution $y = \phi(x)$ on some interval containing x_0. Even if we cannot solve the differential equation and find $\phi(x)$ explicitly, we can still try to find approximate values y_n for $\phi(x_n)$ at a sequence of points

$$x_0, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_3 = x_0 + 3h, \quad \ldots$$

starting at x_0. Here $h > 0$ (or $h < 0$) is called the **step size** of the approximation scheme. In the remainder of this section we will describe three methods for constructing the approximations $\{y_n\}$, namely

1. The Euler method,
2. The improved Euler method, and
3. The fourth-order Runge–Kutta method.

Each of these methods starts with the given value of y_0 and provides a formula for constructing y_{n+1} when you know y_n. The three methods are listed above in increasing order of the complexity of their formulas, but the more complicated formulas produce much better approximations for any given step size h.

The Euler method involves approximating the solution curve $y = \phi(x)$ by a polygonal line (a sequence of straight line segments joined end to end), where each segment has horizontal length h and has slope determined by the value of $f(x, y)$ at the end of the previous segment. Thus, if $x_n = x_0 + nh$, then

$$y_1 = y_0 + f(x_0, y_0)h$$
$$y_2 = y_1 + f(x_1, y_1)h$$
$$y_3 = y_2 + f(x_2, y_2)h$$

and, in general,

Iteration formulas for Euler's method

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + hf(x_n, y_n).$$

Example 6 Use Euler's method to find approximate values for the solution of the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = x - y \\ y(0) = 1 \end{cases}$$

on the interval [0, 1] using

(a) 5 steps of size $h = 0.2$ and

(b) 10 steps of size $h = 0.1$.

Calculate the error at each step, given that the problem (which involves a linear equation, so can be solved explicitly) has solution $y = \phi(x) = x - 1 + 2e^{-x}$.

Solution

(a) Here, we have $f(x, y) = x - y$, $x_0 = 0$, $y_0 = 1$, and $h = 0.2$, so that

$$x_n = \frac{n}{5}, \qquad y_{n+1} = y_n + 0.2(x_n - y_n),$$

and the error is $e_n = \phi(x_n) - y_n$ for $n = 0, 1, 2, 3, 4$, and 5. The results of the calculation, which was done easily using a computer spreadsheet program, are presented in Table 1.

Table 1. Euler approximations with $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}	$e_n = \phi(x_n) - y_n$
0	0.0	1.000000	−1.000000	0.800000	0.000000
1	0.2	0.800000	−0.600000	0.680000	0.037462
2	0.4	0.680000	−0.280000	0.624000	0.060640
3	0.6	0.624000	−0.024000	0.619200	0.073623
4	0.8	0.619200	0.180800	0.655360	0.079458
5	1.0	0.655360	0.344640		0.080399

The exact solution $y = \phi(x)$ and the polygonal line representing the Euler approximation are shown in Figure IV.3. The approximation lies below the solution curve, as is reflected in the positive values in the last column of Table 1, representing the error at each step.

(b) Here, we have $h = 0.1$, so that

$$x_n = \frac{n}{10}, \qquad y_{n+1} = y_n + 0.1(x_n - y_n)$$

for $n = 0, 1, \ldots, 10$. Again we present the results in tabular form:

Table 2. Euler approximations with $h = 0.1$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}	$e_n = \phi(x_n) - y_n$
0	0.0	1.000000	−1.000000	0.900000	0.000000
1	0.1	0.900000	−0.800000	0.820000	0.009675
2	0.2	0.820000	−0.620000	0.758000	0.017462
3	0.3	0.758000	−0.458000	0.712200	0.023636
4	0.4	0.712200	−0.312200	0.680980	0.028440
5	0.5	0.680980	−0.180980	0.662882	0.032081
6	0.6	0.662882	−0.062882	0.656594	0.034741
7	0.7	0.656594	0.043406	0.660934	0.036577
8	0.8	0.660934	0.139066	0.674841	0.037724
9	0.9	0.674841	0.225159	0.697357	0.038298
10	1.0	0.697357	0.302643		0.038402

Observe that the error at the end of the first step is about one-quarter of the error at the end of the first step in part (a), but the final error at $x = 1$ is only about half as large as in part (a). This behaviour is characteristic of Euler's method. ■

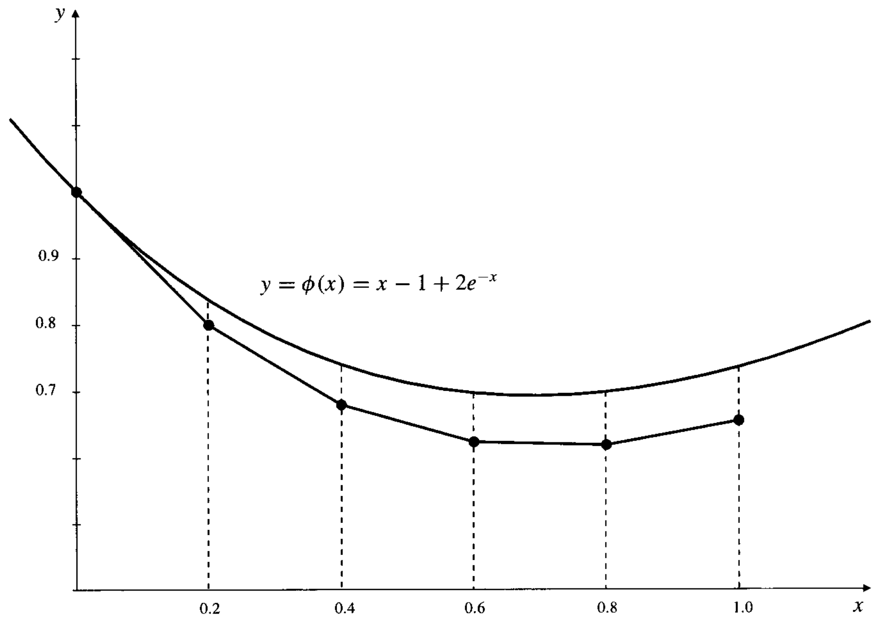


Figure IV.3 The solution $y = \phi(x)$ to $y' = x - y$, $y(0) = 1$, and an Euler approximation to it on $[0, 1]$ with step size $h = 0.2$

If we decrease the step size h, it takes more steps ($n = |x - x_0|/h$) to get from the starting point x_0 to a particular value x where we want to know the value of the solution. For Euler's method it can be shown that the error at each step decreases, on average, proportionally to h^2, but the errors can accumulate from step to step, so the error at x can be expected to decrease proportionally to $nh^2 = |x - x_0|h$. This is consistent with the results of Example 6. Decreasing h and so increasing n is costly in terms of computing resources, so we would like to find ways of reducing the error without decreasing the step size. This is similar to developing better techniques than the Trapezoid Rule for evaluating definite integrals numerically.

The improved Euler method is a step in this direction. The accuracy of the Euler method is hampered by the fact that the slope of each segment in the approximating polygonal line is determined by the value of $f(x, y)$ at one endpoint of the segment. Since f varies along the segment, we would expect to do better by using, say, the average value of $f(x, y)$ at the two ends of the segment, that is, by calculating y_{n+1} from the formula

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2}.$$

Unfortunately, y_{n+1} appears on both sides of this equation, and we can't usually solve the equation for y_{n+1}. We can get around this difficulty by replacing y_{n+1} on the right side by its Euler approximation $y_n + hf(x_n, y_n)$. The resulting formula is the basis for the improved Euler method.

Iteration formulas for the improved Euler method

$$x_{n+1} = x_n + h$$
$$u_{n+1} = y_n + h\,f(x_n, y_n)$$
$$y_{n+1} = y_n + h\,\frac{f(x_n, y_n) + f(x_{n+1}, u_{n+1})}{2}.$$

Example 7 Use the improved Euler method with $h = 0.2$ to find approximate values for the solution to the initial-value problem of Example 6 on $[0, 1]$. Compare the errors with those obtained by the Euler method.

Solution Table 3 summarizes the calculation of five steps of the improved Euler method for $f(x, y) = x - y$, $x_0 = 0$, and $y_0 = 1$.

Table 3. Improved Euler approximations with $h = 0.2$

n	x_n	y_n	u_{n+1}	y_{n+1}	$e_n = \phi(x_n) - y_n$
0	0.0	1.000000	0.800000	0.840000	0.000000
1	0.2	0.840000	0.712000	0.744800	−0.002538
2	0.4	0.744800	0.675840	0.702736	−0.004160
3	0.6	0.702736	0.682189	0.704244	−0.005113
4	0.8	0.704244	0.723395	0.741480	−0.005586
5	1.0	0.741480	0.793184		−0.005721

Observe that the errors are considerably less than 1/10 those obtained in Example 6(a). Of course, more calculations are necessary at each step, but the number of evaluations of $f(x, y)$ required is only twice the number required for Example 6(a). As for numerical integration, if f is complicated, it is these function evaluations that constitute most of the computational "cost" of computing numerical solutions. ■

Remark It can be shown for well-behaved functions f that the error at each step in the improved Euler method is bounded by a multiple of h^3, rather than h^2 as for the (unimproved) Euler method. Thus, the cumulative error at x can be bounded by a constant times $|x - x_0|h^2$. If Example 7 is repeated with 10 steps of size $h = 0.1$, the error at $n = 10$ (i.e., at $x = 1$) is -0.001323, which is about 1/4 the size of the error at $x = 1$ with $h = 0.2$.

The **fourth-order Runge–Kutta method** further improves upon the improved Euler method, but at the expense of requiring more complicated calculations at each step. It requires four evaluations of $f(x, y)$ at each step, but the error at each step is less than a constant times h^5, so the cumulative error decreases like h^4 as h decreases. Like the improved Euler method, this method involves calculating a certain kind of average slope for each segment in the polygonal approximation to the solution to the initial-value problem. We present the appropriate formulas below but cannot derive them here.

Iteration formulas for the Runge–Kutta method

$$\begin{aligned}
x_{n+1} &= x_n + h \\
p_n &= f(x_n, y_n) \\
q_n &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}p_n\right) \\
r_n &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}q_n\right) \\
s_n &= f(x_n + h, y_n + hr_n) \\
y_{n+1} &= y_n + h\,\frac{p_n + 2q_n + 2r_n + s_n}{6}.
\end{aligned}$$

Example 8 Use the fourth-order Runge–Kutta method with $h = 0.2$ to find approximate values for the solution to the initial-value problem of Example 6 on $[0, 1]$. Compare the errors with those obtained by the Euler and improved Euler methods.

Solution Table 4 summarizes the calculation of five steps of the Runge–Kutta method for $f(x, y) = x - y$, $x_0 = 0$, and $y_0 = 1$ according to the formulas above. The table does not show the values of the intermediate quantities p_n, q_n, r_n, and s_n, but columns for these quantities were included in the spreadsheet in which the calculations were made.

Table 4. Fourth-order Runge–Kutta approximations with $h = 0.2$

n	x_n	y_n	$e_n = \phi(x_n) - y_n$
0	0.0	1.000000	0.0000000
1	0.2	0.837467	−0.0000052
2	0.4	0.740649	−0.0000085
3	0.6	0.697634	−0.0000104
4	0.8	0.698669	−0.0000113
5	1.0	0.735770	−0.0000116

The errors here are about 1/500 of the size of the errors obtained with the improved Euler method and about 1/7,000 of the size of the errors obtained with the Euler method. This great improvement was achieved at the expense of doubling the number of function evaluations required in the improved Euler method and quadrupling the number required in the Euler method. If we use 10 steps of size $h = 0.1$ in the Runge–Kutta method, the error at $x = 1$ is reduced to -6.66482×10^{-7}, which is less than 1/16 of its value when $h = 0.2$. ■

Our final example shows what can happen with numerical approximations to a solution that is unbounded.

Example 9 Obtain solutions at $x = 0.4$, $x = 0.8$, and $x = 1.0$ for solutions to the initial-value problem

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

using all three methods described above, and using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ for each method. What do the results suggest about the values of the solution at these points? Compare the results with the actual solution $y = 1/(1-x)$.

Solution The various approximations are calculated using the various formulas described above for $f(x, y) = y^2$, $x_0 = 0$, and $y_0 = 1$. The results are presented in Table 5.

Table 5. Comparing methods and step sizes for $y' = y^2$, $y(0) = 1$

	$h = 0.2$	$h = 0.1$	$h = 0.05$
Euler			
$x = 0.4$	1.488000	1.557797	1.605224
$x = 0.8$	2.676449	3.239652	3.793197
$x = 1.0$	4.109124	6.128898	9.552668
Improved Euler			
$x = 0.4$	1.640092	1.658736	1.664515
$x = 0.8$	4.190396	4.677726	4.897519
$x = 1.0$	11.878846	22.290765	43.114668
Runge–Kutta			
$x = 0.4$	1.666473	1.666653	1.666666
$x = 0.8$	4.965008	4.996628	4.999751
$x = 1.0$	41.016258	81.996399	163.983395

Little useful information can be read from the Euler results. The improved Euler results suggest that the solution exists at $x = 0.4$ and $x = 0.8$, but likely not at $x = 1$. The Runge–Kutta results confirm this and suggest that $y(0.4) = 5/3$ and $y(0.8) = 5$, which are the correct values provided by the actual solution $y = 1/(1-x)$. They also suggest very strongly that the solution "blows up" at (or near) $x = 1$. ■

Exercises

In Exercises 1–10, state the order of the given DE and whether it is linear or nonlinear. If it is linear, is it homogeneous or nonhomogeneous?

1. $\dfrac{dy}{dx} = 5y$

2. $\dfrac{d^2y}{dx^2} + x = y$

3. $y\dfrac{dy}{dx} = x$

4. $y''' + xy' = x \sin x$

5. $y'' + x \sin x\, y' = y$

6. $y'' + 4y' - 3y = 2y^2$

7. $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + t^2 y = t^3$

8. $\cos x \dfrac{dx}{dt} + x \sin t = 0$

9. $y^{(4)} + e^x y'' = x^3 y'$

10. $x^2 y'' + e^x y' = \dfrac{1}{y}$

11. Verify that $y = \cos x$ and $y = \sin x$ are solutions of the DE $y'' + y = 0$. Are any of the following functions solutions: (a) $\sin x - \cos x$, (b) $\sin(x + 3)$, and (c) $\sin 2x$? Justify your answers.

12. Verify that $y = e^x$ and $y = e^{-x}$ are solutions of the DE $y'' - y = 0$. Are any of the following functions solutions: (a) $\cosh x = \frac{1}{2}(e^x + e^{-x})$, (b) $\cos x$, and (c) x^e? Justify your answers.

13. $y_1 = \cos(kx)$ is a solution of $y'' + k^2 y = 0$. Guess and verify another solution y_2 that is not a multiple of y_1. Then find a solution that satisfies $y(\pi/k) = 3$ and $y'(\pi/k) = 3$.

14. $y_1 = e^{kx}$ is a solution of $y'' - k^2 y = 0$. Guess and verify another solution y_2 that is not a multiple of y_1. Then find a solution that satisfies $y(1) = 0$ and $y'(1) = 2$.

15. Find a solution of $y'' + y = 0$ that satisfies $y(\pi/2) = 2y(0)$ and $y(\pi/4) = 3$. *Hint:* see Exercise 11.

16. Find two values of r such that $y = e^{rx}$ is a solution of $y'' - y' - 2y = 0$. Then find a solution of the equation that satisfies $y(0) = 1$, $y'(0) = 2$.

17. Verify that $y = x$ is a solution of $y'' + y = x$, and find a solution y of this DE that satisfies $y(\pi) = 1$ and $y'(\pi) = 0$. *Hint:* use Exercise 11 and Theorem 2.

18. Verify that $y = -e$ is a solution of $y'' - y = e$, and find a solution y of this DE that satisfies $y(1) = 0$ and $y'(1) = 1$. *Hint:* use Exercise 12 and Theorem 2.

Solve the differential equations in Exercises 19–24.

19. $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$

20. $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + 2y^2}$

21. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

22. $\dfrac{dy}{dx} = \dfrac{x^3 + 3xy^2}{3x^2 y + y^3}$

23. $x\dfrac{dy}{dx} = y + x \cos^2\left(\dfrac{y}{x}\right)$

24. $\dfrac{dy}{dx} = \dfrac{y}{x} - e^{-y/x}$

25. Find an equation of the curve in the xy-plane that passes through the point $(1, 3)$ and has, at every point (x, y) on it, slope equal to $1 + (2y/x)$.

26. Show that the change of variables $\xi = x - x_0$, $\eta = y - y_0$ transforms the equation

$$\frac{dy}{dx} = \frac{ax + by + c}{ex + fy + g}$$

into the homogeneous equation

$$\frac{d\eta}{d\xi} = \frac{a\xi + b\eta}{e\xi + f\eta},$$

provided (x_0, y_0) is the solution of the system

$$ax + by + c = 0$$
$$ex + fy + g = 0.$$

27. Use the technique of the previous exercise to solve the equation $\dfrac{dy}{dx} = \dfrac{x + 2y - 4}{2x - y - 3}$.

Show that the DEs in Exercises 28–31 are exact, and solve them.

28. $(xy^2 + y)\,dx + (x^2y + x)\,dy = 0$

29. $(e^x \sin y + 2x)\,dx + (e^x \cos y + 2y)\,dy = 0$

30. $e^{xy}(1 + xy)\,dx + x^2e^{xy}\,dy = 0$

31. $\left(2x + 1 - \dfrac{y^2}{x^2}\right) dx + \dfrac{2y}{x}\,dy = 0$

Show that the DEs in Exercises 32–33 admit integrating factors that are functions of x alone. Then solve the equations.

32. $(x^2 + 2y)\,dx - x\,dy = 0$

33. $(xe^x + x \ln y + y)\,dx + \left(\dfrac{x^2}{y} + x \ln x + x \sin y\right) dy = 0$

34. What condition must the coefficients $M(x, y)$ and $N(x, y)$ satisfy if the equation $M\,dx + N\,dy = 0$ is to have an integrating factor of the form $\mu(y)$, and what DE must the integrating factor satisfy?

35. Find an integrating factor of the form $\mu(y)$ for the equation

$$2y^2(x + y^2)\,dx + xy(x + 6y^2)\,dy = 0,$$

and hence solve the equation. *Hint:* see Exercise 34.

36. Find an integrating factor of the form $\mu(y)$ for the equation $y\,dx - (2x + y^3e^y)\,dy = 0$, and hence solve the equation. *Hint:* see Exercise 34.

37. What condition must the coefficients $M(x, y)$ and $N(x, y)$ satisfy if the equation $M\,dx + N\,dy = 0$ is to have an integrating factor of the form $\mu(xy)$, and what DE must the integrating factor satisfy?

38. Find an integrating factor of the form $\mu(xy)$ for the equation

$$\left(x \cos x + \frac{y^2}{x}\right) dx - \left(\frac{x \sin x}{y} + y\right) dy = 0,$$

and hence solve the equation. *Hint:* see Exercise 37.

A computer is almost essential for doing Exercises 39–44. The calculations are easily done with a spreadsheet program in which formulas for calculating the various quantities involved can be replicated down columns to automate the iteration process.

39. Use the Euler method with step sizes (a) $h = 0.2$, (b) $h = 0.1$, and (c) $h = 0.05$ to approximate $y(2)$ given that $y' = x + y$ and $y(1) = 0$.

40. Repeat Exercise 39 using the improved Euler method.

41. Repeat Exercise 39 using the Runge–Kutta method.

42. Use the Euler method with step sizes (a) $h = 0.2$, and (b) $h = 0.1$ to approximate $y(2)$ given that $y' = xe^{-y}$ and $y(0) = 0$.

43. Repeat Exercise 42 using the improved Euler method.

44. Repeat Exercise 42 using the Runge–Kutta method.

Solve the integral equations in Exercises 45–46 by rephrasing them as initial-value problems.

45. $y(x) = 2 + \displaystyle\int_1^x \big(y(t)\big)^2\,dt$. *Hint:* find $\dfrac{dy}{dx}$ and $y(1)$.

46. $u(x) = 1 + 3\displaystyle\int_2^x t^2u(t)\,dt$. *Hint:* find $\dfrac{du}{dx}$ and $u(2)$.

47. The methods of this section can be used to approximate definite integrals numerically. For example,

$$I = \int_a^b f(x)\,dx$$

is given by $I = y(b)$, where

$$y' = f(x) \qquad \text{and} \qquad y(a) = 0.$$

Show that one step of the Runge–Kutta method with $h = b - a$ gives the same result for I as does Simpson's Rule with two subintervals of length $h/2$.

48. If $\phi(0) = A \geq 0$ and $\phi'(x) \geq k\phi(x)$ on $[0, X]$, where $k > 0$ and $X > 0$ are constants, show that $\phi(x) \geq Ae^{kx}$ on $[0, X]$. *Hint:* calculate $(d/dx)(\phi(x)/e^{kx})$.

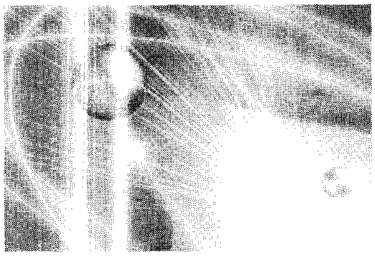
* **49.** Consider the three initial-value problems

(A)	$u' = u^2$	$u(0) = 1$
(B)	$y' = x + y^2$	$y(0) = 1$
(C)	$v' = 1 + v^2$	$v(0) = 1$

(a) Show that the solution of (B) remains between the solutions of (A) and (C) on any interval $[0, X]$ where solutions of all three problems exist. *Hint:* we must have $u(x) \geq 1$, $y(x) \geq 1$, and $v(x) \geq 1$ on $[0, X]$. (Why?) Apply the result of Exercise 48 to $\phi = y - u$ and to $\phi = v - y$.

(b) Find explicit solutions for problems (A) and (C). What can you conclude about the solution to problem (B).

(c) Use the Runge–Kutta method with $h = 0.05$, $h = 0.02$, and $h = 0.01$ to approximate the solution to (B) on $[0, 1]$. What can you conclude now?



Appendix V

Doing Calculus with Maple

Computer algebra systems like Maple and Mathematica are capable of doing most of the tedious calculations involved in doing calculus, especially the very intensive calculations required by many applied problems. (They cannot, of course, do the thinking for you; you must still fully understand what you are doing and what are the limitations of such programs.) Throughout this text we have inserted material illustrating how to use **Maple** to do common calculus-oriented calculations. These insertions range in length from single paragraphs and remarks to entire sections. To help you locate the Maple material appropriate for specific topics, we include below a list pointing to the text sections containing Maple examples and the pages on which they start.

Note, however, that this material assumes you are familiar with the basics of starting a Maple session, preferably with a graphical user interface which typically displays the prompt ">" when it is waiting for your input. In this book the input is shown in colour. It normally concludes with a semicolon ";" followed by pressing the `<enter>` key, which we omit from our examples. The output is typically printed by Maple centred in the window; we show it in black. For instance,

```
> factor(x^2-x-2);
```

$$(x+1)(x-2)$$

The author used Maple V, Release 5, and Maple 6 for preparing these examples. Some of the examples involve procedure definition and worksheet files available from the website for this text:

```
http://www.pearsoned.ca/text/adams_calc
```

Two of the Maple procedures used in Section 13.7 for finding roots of systems of nonlinear equations and for finding and classifying critical points of functions of several variables are quite lengthy, and rather than list them there, we have included their definitions later on in this Appendix.

The Maple examples in this book are by no means complete or exhaustive. For a more complete treatment of Maple as a tool for doing calculus, the author highly recommends the excellent Maple lab manual *Calculus: The Maple Way* written by his colleague, Professor Robert Israel of the University of British Columbia. Like this book, it is published by Pearson Canada under the Addison-Wesley logo.

List of Maple Examples and Discussion

Several of the topics in the above list are covered over several pages. Only the first page is listed.

The "newtroot" Procedure of Section 13.7

Here is a listing of the Maple procedures `newtroot` discussed in Section 13.7. You can learn much about Maple by reading this listing and trying to understand what it is doing.

```
newtroot:=proc(F::procedure,v,m::integer,tol::float)
local i,j,k,v0,v1,w,FV,JF,A,b,error,n;
error := tol + 1.0; # tol = desired accuracy
i := 1; v0 := v;
if type(v,list) then
    convert(v0,vector);
```

```
     n := vectdim(v0);
   else n := 1 fi;
w := vector(n);
if n = 1 then
     while tol < error and i < m +1 do
       v1:= evalf(v0-F(v0)/(D(F)(v0))); #Newton iteration
       error := abs(v1-v0);
       v0 := v1; #v0 becomes the new approximation
       print(i, v0, F(v0), error);
       i:= i+1;
     od
   else FV := v -> F(seq(v[j],j=1..n));
     JF := proc(FF::procedure,vv::vector)
       jacobian(FF(vv),vv);
     end;
     while tol < error and i < m +1 do
       A:= subs(seq(w[k]=v0[k],k=1..n),JF(FV,w));
              # A = JF(v0).
       b:= FV(v0);
       v1 := evalf(evalm(v0 - linsolve(A,b)));
              #Newton iteration
       error := norm(v0-v1);
       v0 := evalm(v1);
              #v0 becomes the new approximation
       print(i, v0,FV(v0),error);
       i := i + 1;
     od
  fi;
if error <= tol then
     RETURN(evalm(v0))
   else print('FAILED_TO_FIND_ROOT',error);
     RETURN(evalm(v0))
   fi;
end;
```

The scalar case $n = 1$ evidently calculates the Newton's Method approximation v_1 from v_0 and then renames v_0 to be this new approximation before doing another iteration. So does the vector case $n > 1$, handled by the "else" clause. However, it does not use determinants (i.e., Cramer's Rule) to calculate the next approximation as we did in Section 13.6. Instead, it uses the **Jacobian matrix** $\mathcal{A} = JF(\mathbf{v}_0)$ of $\mathbf{F}$ at $\mathbf{v}_0$. The next approximation $\mathbf{v}_1$ satisfies the system of equations

$$\mathcal{A}(\mathbf{v}_1 - \mathbf{v}_0) + \mathbf{b} = \mathbf{0}, \qquad \text{where} \quad \mathbf{b} = \mathbf{F}(\mathbf{v}_0).$$

The Maple function `linsolve(A,b)` determines the solution $\mathbf{x} = \mathcal{A}^{-1}\mathbf{b}$ of the system $\mathcal{A}\mathbf{x} = \mathbf{b}$, so the procedure calculates $\mathbf{v}_1 = \mathbf{v}_0 - \mathcal{A}^{-1}\mathbf{b}$. The `evalm` and `evalf` operators then convert the resulting list of solutions to a vector with real components.

The "newtcp" Procedure of Section 13.7

This procedure, also discussed in Section 13.7, is a variant of `newtroot` used to find and classify the critical points of a function of several variables.

```
newtcp:=proc(F::procedure,v,m::integer,tol::float)
local i,j,k,v0,v1,w,FV,GRADF,HF,A,b,error,n;
error := tol + 1.0; # tol = desired accuracy
i := 1; v0 := v;
if type(v,list) then
    convert(v0,vector);
    n := vectdim(v0);
  else n := 1 fi;
w := vector(n);
if n = 1 then
    while tol < error and i < m +1 do
      v1:= evalf(v0-D(F)(v0)/(D@@2)(F)(v0));
      error := abs(v1-v0);
      v0 := v1;
      print(i,v0,F(v0),error);
      i:= i+1;
    od;
  else
    FV := v -> F(seq(v[j],j=1..n));
    GRADF := proc(FF::procedure,vv::vector)
      grad(FF(vv),vv);
    end;
    HF := proc(FF::procedure,vv::vector)
      hessian(FF(vv),vv);
    end;
    while tol < error and i < m +1 do
      A:= subs(seq(w[k]=v0[k],k=1..n),HF(FV,w));
                #A=HF(v0).
      b:= subs(seq(w[k]=v0[k],k=1..n),GRADF(FV,w));
      v1 := evalf(evalm(v0 - linsolve(A,b)));
                #Newton iteration
      error := norm(v0-v1);
      v0 := evalm(v1);
                #v0 becomes the new approximation
      print(i,v0,FV(v0),error);
      i := i + 1;
    od
  fi;
if (error <= tol) then
    if (n=1) then
      print('Second_deriv',evalf((D@@2)(F)(v0)));
      RETURN(evalm(v0),F(v0))
    else print('Eigenvalues',evalf(eigenvals(A))) fi;
    RETURN(evalm(v0),FV(v0))
  else print('FAILED',error);
    RETURN(evalm(v0))
fi;
```

```
end;
```

Answers to Odd-Numbered Exercises

Chapter P

Preliminaries

Section P.1 (page 11)

1. $0.\overline{2}$ **3.** $4/33$

5. $1/7 = 0.\overline{142857}$, $2/7 = 0.\overline{285714}$, $3/7 = 0.\overline{428571}$, $4/7 = 0.\overline{571428}$, $5/7 = 0.\overline{714285}$, $6/7 = 0.\overline{857142}$

7. $[0, 5]$ **9.** $]-\infty, -6[\,\cup\,]-5, \infty[$

11. $]-2, \infty[$ **13.** $]-\infty, -2[$

15. $(-\infty, 5/4]$ **17.** $]0, \infty[$

19. $]-\infty, 5/3[\,\cup\,]2, \infty[$ **21.** $[0, 2]$

23. $]-2, 0[\,\cup\,]2, \infty[$ **25.** $[-2, 0[\,\cup\,[4, \infty[$

27. $x = -3,\ 3$ **29.** $t = -1/2,\ -9/2$

31. $s = -1/3,\ 17/3$ **33.** $(-2, 2)$

35. $[-1, 3]$ **37.** $\left]\dfrac{5}{3}, 3\right[$

39. $[0, 4]$ **41.** $x > 1$

43. true if $a \geq 0$, false if $a < 0$

Section P.2 (page 18)

1. $\Delta x = 4$, $\Delta y = -3$, dist $= 5$

3. $\Delta x = -4$, $\Delta y = -4$, dist $= 4\sqrt{2}$

5. $(2, -4)$

7. circle, centre $(0, 0)$, radius 1

9. points inside and on circle, centre $(0, 0)$, radius 1

11. points on and above the parabola $y = x^2$

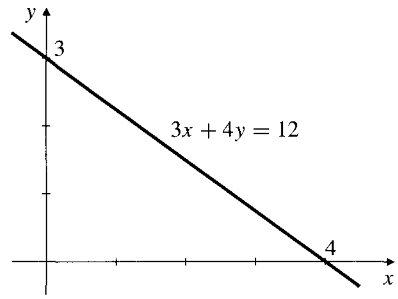
13. (a) $x = -2$, (b) $y = 5/3$

15. $y = x + 2$ **17.** $y = 2x + b$

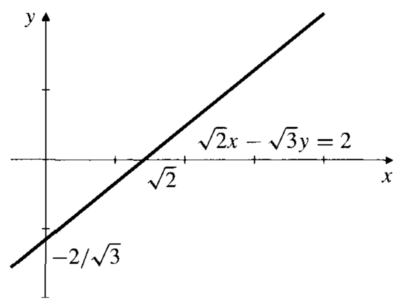
19. above **21.** $y = 3x/2$

23. $y = (7 - x)/3$ **25.** $y = \sqrt{2} - 2x$

27. 4, 3,



29. $\sqrt{2}$, $-2/\sqrt{3}$



31. (a) $y = x - 1$, (b) $y = -x + 3$

33. $(2, -3)$ **37.** 5

39. \$23,000 **43.** $(-2, -2)$

45. $\left(\frac{1}{3}(x_1 + 2x_2), \frac{1}{3}(y_1 + 2y_2)\right)$

47. circle, centre $(2, 0)$, radius 4

49. perp. if $k = -8$, parallel if $k = 1/2$

Section P.3 (page 25)

1. $x^2 + y^2 = 16$ **3.** $x^2 + y^2 + 4x = 5$

5. $(1, 0)$, 2 **7.** $(1, -2)$, 3

9. exterior of circle, centre $(0, 0)$, radius 1

11. closed disk, centre $(-1, 0)$, radius 2

13. washer shaped region between the circles of radius 1 and 2 centred at $(0, 0)$

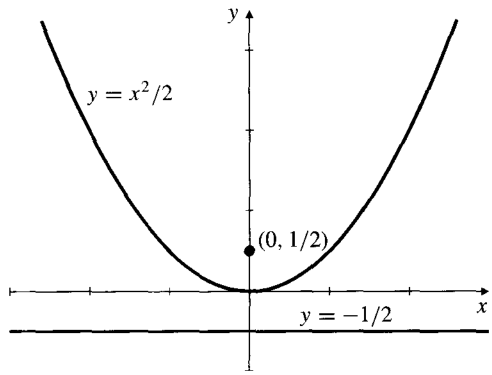
15. first octant region lying inside the two circles of radius 1 having centres at $(1, 0)$ and $(0, 1)$

17. $x^2 + y^2 + 2x - 4y < 1$

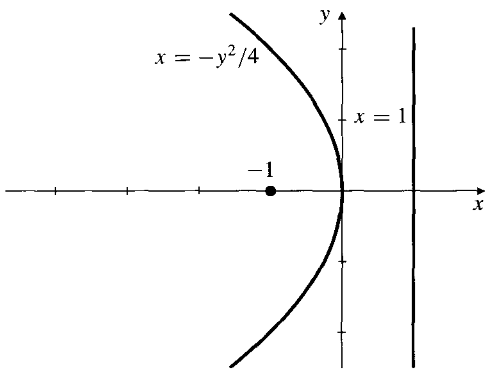
19. $x^2 + y^2 < 2$, $x \geq 1$ **21.** $x^2 = 16y$

23. $y^2 = 8x$

25. $(0, 1/2)$, $y = -1/2$



27. $(-1, 0)$, $x = 1$



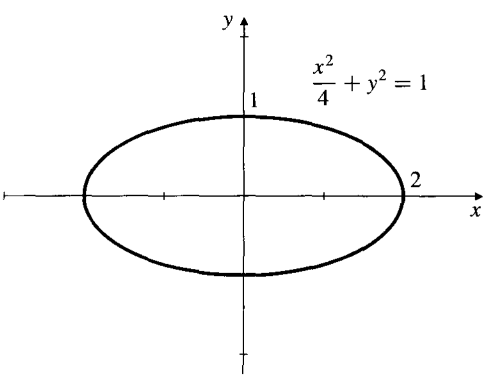
29. (a) $y = x^2 - 3$, (b) $y = (x-4)^2$, (c) $y = (x-3)^2 + 3$, (d) $y = (x-4)^2 - 2$

31. $y = \sqrt{(x/3) + 1}$ **33.** $y = \sqrt{(3x/2) + 1}$

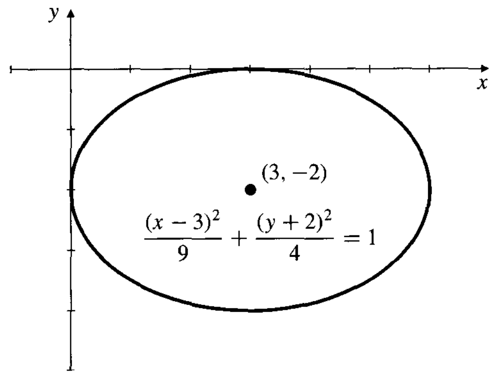
35. $y = -(x+1)^2$ **37.** $y = (x-2)^2 - 2$

39. $(2, 7)$, $(1, 4)$ **41.** $(4, -3)$, $(-4, 3)$

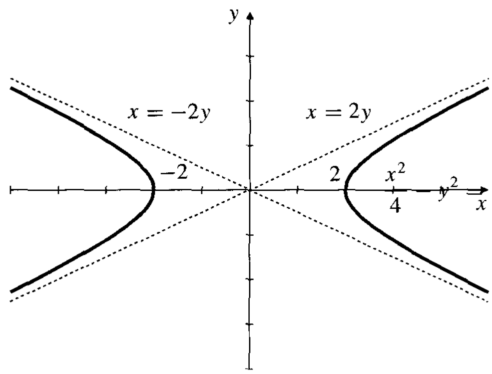
43. ellipse, centre $(0, 0)$, semiaxes 2, 1



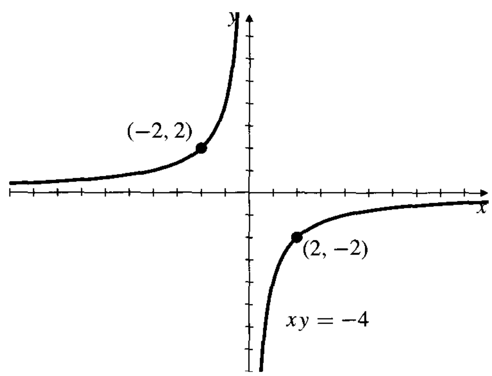
45. ellipse, centre $(3, -2)$, semiaxes 3, 2



47. hyperbola, centre $(0, 0)$, asymptotes $x = \pm 2y$, vertices $(\pm 2, 0)$

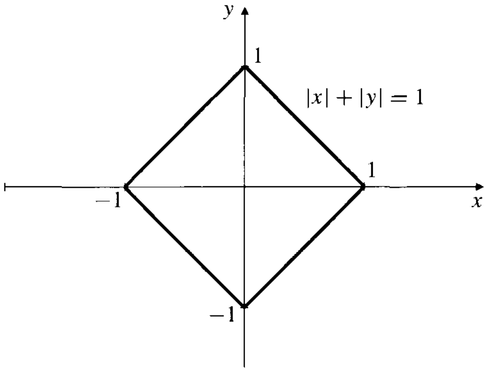


49. rectangular hyperbola, asymptotes $x = 0$ and $y = 0$, vertices $(2, -2)$ and $(-2, 2)$



51. (a) reflecting the graph in the y-axis, (b) reflecting the graph in the x-axis.

53.



Section P.4 (page 35)

1. $\mathcal{D}(f) = \mathbb{R}$, $\mathcal{R}(f) = [1, \infty[$

3. $\mathcal{D}(G) =]-\infty, 4]$, $\mathcal{R}(g) = [0, \infty[$

5. $\mathcal{D}(h) =]-\infty, 2[$, $\mathcal{R}(h) =]-\infty, \infty[$

7. Only (ii) is the graph of a function. Vertical lines can meet the others more than once.

11. even, sym. about y-axis

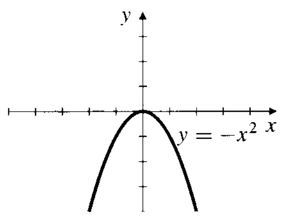
13. odd, sym. about (0, 0) **15.** sym. about (2, 0)

17. sym. about $x = 3$

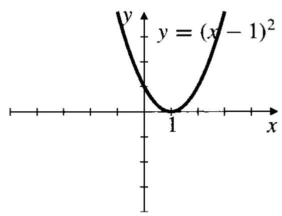
19. even, sym. about y-axis

21. no symmetry

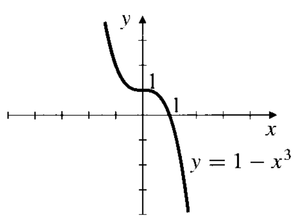
23.



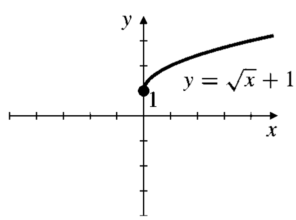
25.



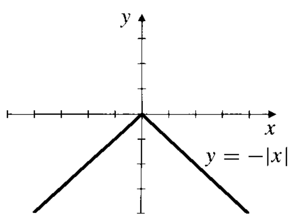
27.



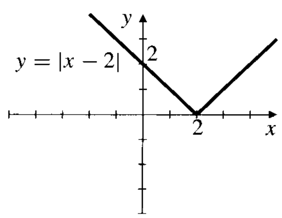
29.



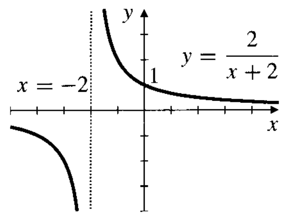
31.



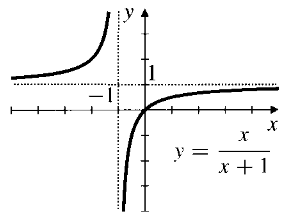
33.

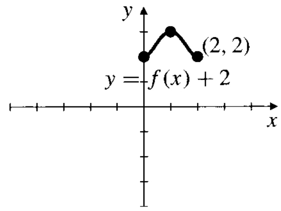


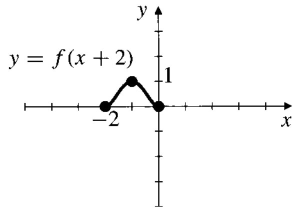
35.



37.



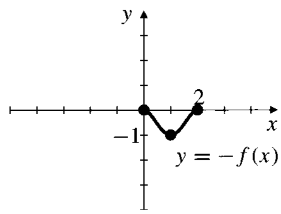


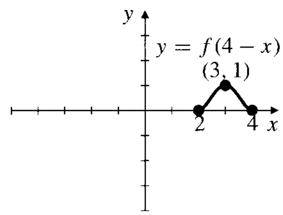


39. $D = [0, 2]$, $R = [2, 3]$

41. $D = [-2, 0]$, $R = [0, 1]$

43. $D = [0, 2]$, $R = [-1, 0]$





45. $D = [2, 4]$, $R = [0, 1]$ **47.** $[-0.18, 0.68]$

49. $y = 3/2$

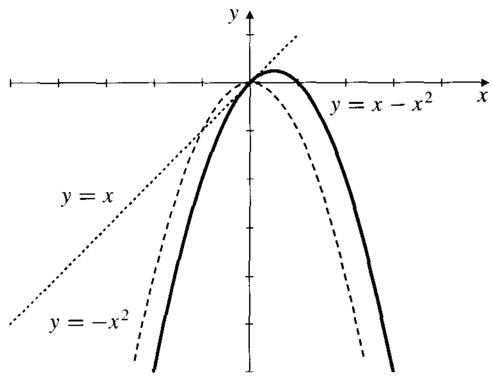
51. $(2, 1)$, $y = x - 1$, $y = 3 - x$

53. $f(x) = 0$

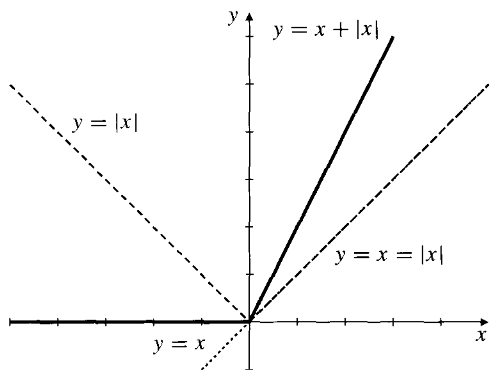
Section P.5 (page 41)

1. The domains of $f+g$, $f-g$, fg, and g/f are $[1, \infty[$. The domain of f/g is $]1, \infty[$.
$(f + g)(x) = x + \sqrt{x - 1}$
$(f - g)(x) = x - \sqrt{x - 1}$
$(fg)(x) = x\sqrt{x - 1}$
$(f/g)(x) = x/\sqrt{x - 1}$
$(g/f)(x) = \sqrt{x - 1}/x$

3.



5.



7. (a) 2, (b) 22, (c) $x^2 + 2$, (d) $x^2 + 10x + 22$, (e) 5, (f) -2, (g) $x + 10$, (h) $x^4 - 6x^2 + 6$

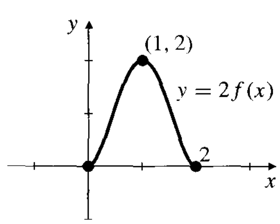
9. (a) $(x - 1)/x$, $x \neq 0$, 1,
(b) $1/(1 - \sqrt{x - 1})$ on $[1, 2[\cup]2, \infty[$,
(c) $\sqrt{x/(1 - x)}$, on $[0, 1[$
(d) $\sqrt{\sqrt{x - 1} - 1}$, on $[2, \infty[$

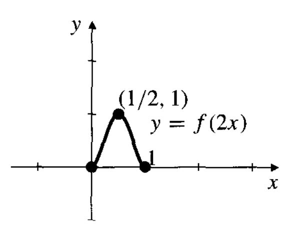
11. $(x + 1)^2$

13. x^2

15. $1/(x - 1)$

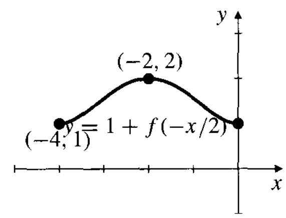
19. $D = [0, 2]$, $R = [0, 2]$





21. $D = [0, 1]$, $R = [0, 1]$

23. $D = [-4, 0]$, $R = [1, 2]$

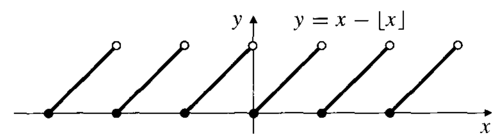


25.

27. (a) $A = 0$, B arbitrary, or $A = 1$, $B = 0$
(b) $A = -1$, B arbitrary, or $A = 1$, $B = 0$

29. all integers

31.



33. f^2, g^2, $f \circ f$, $f \circ g$, $g \circ f$ are even
fg, f/g, g/f, $g \circ g$ are odd
$f + g$ is neither, unless either $f(x) = 0$ or $g(x) = 0$.

Section P.6 (page 55)

1. $-1/\sqrt{2}$

3. $\sqrt{3}/2$

5. $(\sqrt{3} - 1)/(2\sqrt{2})$

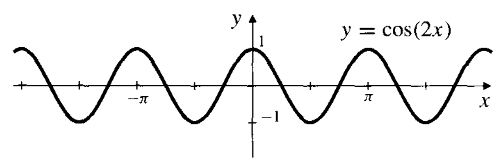
7. $-\cos x$

9. $-\cos x$

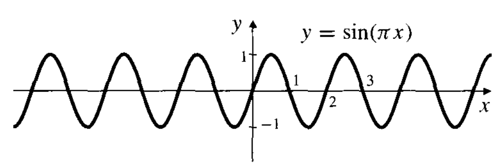
11. $1/(\sin x \cos x)$

17. $3 \sin x - 4 \sin^3 x$

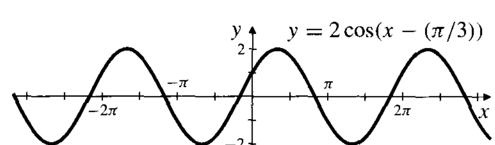
19. period π



21. period 2



23.



25. $\cos\theta = -4/5$, $\tan\theta = -3/4$

27. $\sin\theta = -2\sqrt{2}/3$, $\tan\theta = -2\sqrt{2}$

29. $\cos\theta = -\sqrt{3}/2$, $\tan\theta = 1/\sqrt{3}$

31. $a = 1,\ b = \sqrt{3}$
33. $b = 5/\sqrt{3},\ c = 10/\sqrt{3}$
35. $a = b \tan A$
37. $a = b \cot B$
39. $c = b \sec A$
41. $\sin A = \sqrt{c^2 - b^2}/c$
43. $\sin B = 3/(4\sqrt{2})$
45. $\sin B = \sqrt{135}/16$
47. $6/(1 + \sqrt{3})$
49. $b = 4 \sin 40° / \sin 70° \approx 2.736$
51. approx. 16.98 m

Chapter 1
Limits and Continuity

Section 1.1 (page 61)

1. $((t+h)^2 - t^2)/h$ m/s
3. 4 m/s
5. −3 m/s, 3 m/s, 0 m/s
7. to the left, stopped, to the right
9. height 2, moving down
11. −1 ft/s, weight moving downward
13. day 45

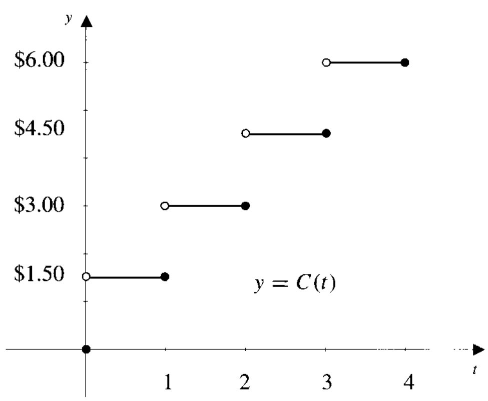
Section 1.2 (page 70)

1. (a) 1, (b) 0, (c) 1
3. 1
5. 0
7. 1
9. 2/3
11. 0
13. 0
15. does not exist
17. 1/6
19. 0
21. −1
23. does not exist
25. 2
27. 3/8
29. −1/2
31. 8/3
33. 1/4
35. $1/\sqrt{2}$
37. $2x$
39. $-1/x^2$
41. $1/(2\sqrt{x})$
43. 1
45. 1/2
47. 1
49. 0
51. 2
53. does not exist
55. does not exist
57. $-1/(2a)$
59. 0
61. −2
63. π^2
65. (a) 0, (b) 8, (c) 9, (d) −3
67. 5
69. 1
71. 0.7071
73. $\lim_{x\to 0} f(x) = 0$
75. 2
77. $x^{1/3} < x^3$ on $]-1, 0[$ and $]1, \infty[$,
$x^{1/3} > x^3$ on $]-\infty, -1[$ and $]0, 1[$,
$\lim_{x\to a} h(x) = a$ for $a = -1, 0,$ and 1

Section 1.3 (page 77)

1. 1/2
3. −3/5
5. 0
7. −3
9. $-2/\sqrt{3}$
11. does not exist
13. $+\infty$
15. 0
17. $-\infty$
19. $-\infty$
21. ∞
23. $-\infty$
25. ∞
27. $-\sqrt{2}/4$
29. −2
31. −1
33. horiz: $y = 0,\ y = -1$, vert: $x = 0$
35. 1
37. 1
39. $-\infty$
41. 2
43. −1
45. 1
47. 3
49. does not exist
51. 1
53. $C(t)$ has a limit at every real t except at the integers. $\lim_{t\to t_0-} C(t) = C(t_0)$ everywhere, but

$$\lim_{t\to t_0+} C(t) = \begin{cases} C(t_0) & \text{if } t_0 \text{ not integral} \\ C(t_0) + 1.5 & \text{if } t_0 \text{ an integer} \end{cases}$$



55. (a) B, (b) A, (c) A, (d) A

Section 1.4 (page 87)

1. at −2, right cont. and cont., at −1 disc., at 0 disc. but left cont., at 1 disc. and right cont., at 2 disc.
3. no abs. max, abs. min 0
5. no
7. cont. everywhere
9. cont. everywhere except at $x = 0$, disc. at $x = 0$
11. cont. everywhere except at the integers, discontinuous but left-continuous at the integers

13. 4, $x+2$

15. 1/5, $(t-2)/(t+2)$

17. $k=8$

19. no max, min = 0

21. 16

23. 5

25. f positive on $]-1,0[$ and $]1,\infty[$; f negative on $]-\infty,-1[$ and $]0,1[$

27. f positive on $]-\infty,-2[$, $]-1,1[$ and $]2,\infty[$; f negative on $]-2,-1[$ and $]1,2[$

35. max 1.593 at -0.831, min -0.756 at 0.629

37. max $31/3 \approx 10.333$ at $x=3$, min 4.762 at $x=1.260$

39. 0.682

41. -0.6367326508, 1.409624004

Section 1.5 (page 94)

1. between 12°C and 20°C

3. (1.99, 2.01)

5. (0.81, 1.21)

7. $\delta=0.01$

9. $\delta \approx 0.0165$

Review Exercises (page 95)

1. 13

3. 12

5. 4

7. does not exist

9. does not exist

11. $-\infty$

13. $12\sqrt{3}$

15. 0

17. does not exist

19. $-1/3$

21. $-\infty$

23. ∞

25. does not exist

27. 0

29. 2

31. no disc.

33. disc. and left cont. at 2

35. disc. and right cont. at $x=1$

37. no disc.

Challenging Problems (page 96)

1. to the right

3. $-1/4$

5. 3

7. T, F, T, F, F

Chapter 2 Differentiation

Section 2.1 (page 102)

1. $y=3x-1$

3. $y=8x-13$

5. $y=12x+24$

7. $x-4y=-5$

9. $x-4y=-2$

11. $y=2x_0x-x_0^2$

13. no

15. yes, $x=-2$

17. yes, $x=0$

19. (a) $3a^2$; (b) $y=3x-2$ and $y=3x+2$

21. (1, 1), $(-1,1)$

23. $k=3/4$

25. horiz. tangent at (0, 0), (3, 108), (5, 0)

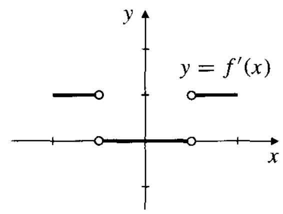
27. horiz. tangent at $(-0.5, 1.25)$, no tangents at $(-1,1)$ and $(1,-1)$

29. horiz. tangent at $(0,-1)$

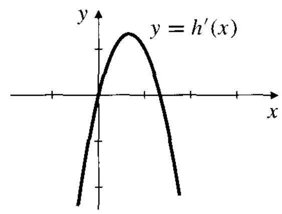
31. no, consider $y=x^{2/3}$ at (0, 0)

Section 2.2 (page 110)

1.



3.



5. on $[-2,2]$ except at $x=-1$ and $x=1$

7. slope positive for $x<1.5$, negative for $x>1.5$; horizontal tangent at $x=1.5$

9. singular points at $x=-1$, 0, 1, horizontal tangents at about $x=\pm 0.57$

11. $2x-3$

13. $3x^2$

15. $\dfrac{1}{\sqrt{2t+1}}$

17. $1-\dfrac{1}{x^2}$

19. $-\dfrac{x}{(1+x^2)^{3/2}}$

21. $-\dfrac{1}{2(1+x)^{3/2}}$

23. Define $f(0)=0$, f is not differentiable at 0

25. at $x=-1$ and $x=-2$

27.

x	$\dfrac{f(x)-f(2)}{x-2}$
1.9	−0.26316
1.99	−0.25126
1.999	−0.25013
1.9999	−0.25001

x	$\dfrac{f(x)-f(2)}{x-2}$
2.1	−0.23810
2.01	−0.24876
2.001	−0.24988
2.0001	−0.24999

$$\frac{d}{dx}\left(\frac{1}{x}\right)\bigg|_{x=2} = -\frac{1}{4}$$

29. $x-6y=-15$

31. $y=\dfrac{2}{a^2+a}-\dfrac{2(2a+1)}{(a^2+a)^2}(t-a)$

33. $22t^{21}$, all t

35. $-(1/3)x^{-4/3}$, $x \neq 0$

37. $(119/4)s^{115/4}$, $s \geq 0$

39. -16

41. $1/(8\sqrt{2})$

43. $y=a^2x-a^3+\dfrac{1}{a}$

45. $y=6x-9$ and $y=-2x-1$

47. $\dfrac{1}{2\sqrt{2}}$

51. $f'(x)=\frac{1}{3}x^{-2/3}$

Section 2.3 (page 119)

1. $6x-5$ **3.** $2Ax+B$

5. $\frac{1}{3}s^4-\frac{1}{5}s^2$

7. $\frac{1}{3}t^{-2/3}+\frac{1}{2}t^{-3/4}+\frac{3}{5}t^{-4/5}$

9. $x^{2/3}+x^{-8/5}$ **11.** $\dfrac{5}{2\sqrt{x}}-\frac{3}{2}\sqrt{x}-\frac{5}{6}x^{3/2}$

13. $-\dfrac{2x+5}{(x^2+5x)^2}$ **15.** $\dfrac{\pi^2}{(2-\pi t)^2}$

17. $(4x^2-3)/x^4$

19. $-t^{-3/2}+(1/2)t^{-1/2}+(3/2)\sqrt{t}$

21. $-\dfrac{24}{(3+4x)^2}$ **23.** $\dfrac{1}{\sqrt{t}(1-\sqrt{t})^2}$

25. $\dfrac{ad-bc}{(cx+d)^2}$

27. $10+70x+150x^2+96x^3$

29. $2x(\sqrt{x}+1)(5x^{2/3}-2)+\dfrac{1}{2\sqrt{x}}(x^2+4)(5x^{2/3}-2)$
$+\dfrac{10}{3}x^{-1/3}(x^2+4)(\sqrt{x}+1)$

31. $\dfrac{6x+1}{(6x^2+2x+1)^2}$ **33.** -1

35. 20 **37.** $-\dfrac{1}{2}$

39. $-\dfrac{1}{18\sqrt{2}}$ **41.** $y=4x-6$

43. $(1,2)$ and $(-1,-2)$ **45.** $\left(-\frac{1}{2},\frac{4}{3}\right)$

47. $y=b-\dfrac{b^2x}{4}$

49. $y=12x-16,\ y=3x+2$

51. $x/\sqrt{x^2+1}$

Section 2.4 (page 125)

1. $12(2x+3)^5$ **3.** $-20x(4-x^2)^9$

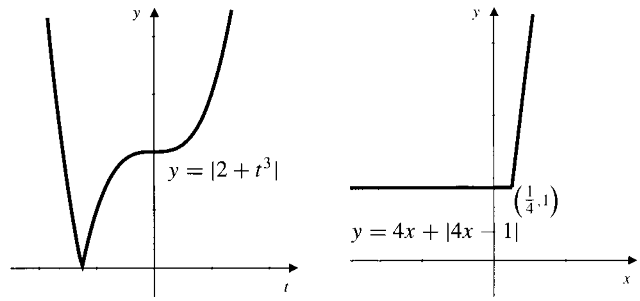
5. $\dfrac{30}{t^2}\left(2+\dfrac{3}{t}\right)^{-11}$ **7.** $\dfrac{12}{(5-4x)^2}$

9. $-2x\,\mathrm{sgn}\,(1-x^2)$ **11.** $\begin{cases}8 & \text{if } x>1/4\\ 0 & \text{if } x<1/4\end{cases}$

13. $\dfrac{-3}{2\sqrt{3x+4}(2+\sqrt{3x+4})^2}$

15. $-\dfrac{5}{3}\left(1-\dfrac{1}{(u-1)^2}\right)\left(u+\dfrac{1}{u-1}\right)^{-8/3}$

17.



23. $(5-2x)f'(5x-x^2)$ **25.** $\dfrac{f'(x)}{\sqrt{3+2f(x)}}$

27. $\dfrac{1}{\sqrt{x}}f'(3+2\sqrt{x})$

29. $15f'(4-5t)f'(2-3f(4-5t))$

31. $\dfrac{3}{2\sqrt{2}}$ **33.** 102

35. $-6\left(1-\frac{15}{2}(3x)^4\left((3x)^5-2\right)^{-3/2}\right)$
$\times\left(x+\left((3x)^5-2\right)^{-1/2}\right)^{-7}$

37. $y=2^{3/2}-\sqrt{2}(x+1)$ **39.** $y=\frac{1}{27}+\frac{5}{162}(x+2)$

41. $\dfrac{x(x^4+2x^2-2)}{(x^2+1)^{5/2}}$ **43.** 857,592

45. no; yes; both functions are equal to x^2.

Section 2.5 (page 131)

3. $-3\sin 3x$ **5.** $\pi\sec^2\pi x$

7. $3\csc^2(4-3x)$ **9.** $r\sin(s-rx)$

11. $2\pi x\cos(\pi x^2)$ **13.** $\dfrac{-\sin x}{2\sqrt{1+\cos x}}$

15. $-(1+\cos x)\sin(x+\sin x)$

17. $(3\pi/2)\sin^2(\pi x/2)\cos(\pi x/2)$

19. $a\cos 2at$ **21.** $2\cos(2x)+2\sin(2x)$

23. $\sec^2 x-\csc^2 x$ **25.** $\tan^2 x$

27. $-t\sin t$ **29.** $1/(1+\cos x)$

31. $2x\cos(3x)-3x^2\sin(3x)$

33. $2x[\sec(x^2)\tan^2(x^2)+\sec^3(x^2)]$

35. $-\sec^2 t\sin(\tan t)\cos(\cos(\tan t))$

39. $y=\pi-x,\ \ y=x-\pi$

41. $y=1-(x-\pi)/4,\ \ y=1+4(x-\pi)$

43. $y=\dfrac{1}{\sqrt{2}}+\dfrac{\pi}{180\sqrt{2}}(x-45)$

45. $\pm(\pi/4,1)$ **49.** yes, (π,π)

51. yes, $(2\pi/3,(2\pi/3)+\sqrt{3})$, $(4\pi/3,(4\pi/3)-\sqrt{3})$

53. 2 **55.** 1

57. 1/2

59. infinitely many, 0.336508, 0.161228

Section 2.6 (page 139)

1. $c = \dfrac{a+b}{2}$

3. $c = \pm\dfrac{2}{\sqrt{3}}$

9. inc. on $\left]-\infty, -\dfrac{2}{\sqrt{3}}\right[$ and $\left]\dfrac{2}{\sqrt{3}}, \infty\right[$, dec. on $\left]-\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}\right[$

11. inc. on $]-2, 0[$ and $]2, \infty[$; dec. on $]-\infty, -2[$ and $]0, 2[$

13. inc. on $]-\infty, 3[$ and $]5, \infty[$; dec. on $]3, 5[$

15. inc. on $]-\infty, \infty[$

17. The two separate applications of MVT cannot be expected to give the same value of c.

Section 2.7 (page 145)

1. 4%

3. −4%

5. 1%

7. 6%

9. 8 ft^2/ft

11. $1/\sqrt{\pi A}$ units/square unit

13. 16π m^3/m

15. $\dfrac{dC}{dA} = \sqrt{\dfrac{\pi}{A}}$ length units/area unit

17. CP. $x = 0$, incr. $x > 0$, decr. $x < 0$

19. CP. $x = 0$, $x = -4$, incr. on $]-\infty, -4[$ and $]0, \infty[$, decr. on $]-4, 0[$

23. 0.535898, 7.464102

25. 0, -0.518784

27. (a) 10,500 L/min, 3,500 L/min, (b) 7,000 L/min

29. decreases at 1/8 lb/mi

31. (a) \$300, (b) $C(101) - C(100) =$ \$299.50

33. (a) −\$2.00, (b) \$9.11

Section 2.8 (page 150)

1. $\begin{cases} y' = -14(3-2x)^6, \\ y'' = 168(3-2x)^5, \\ y''' = -1680(3-2x)^4 \end{cases}$

3. $\begin{cases} y' = -12(x-1)^{-3}, \\ y'' = 36(x-1)^{-4}, \\ y''' = -144(x-1)^{-5} \end{cases}$

5. $\begin{cases} y' = \frac{1}{3}x^{-2/3} + \frac{1}{3}x^{-4/3}, \\ y'' = -\frac{2}{9}x^{-5/3} - \frac{4}{9}x^{-7/3} \\ y''' = \frac{10}{27}x^{-8/3} + \frac{28}{27}x^{-10/3} \end{cases}$

7. $\begin{cases} y' = \frac{5}{2}x^{3/2} + \frac{3}{2}x^{-1/2} \\ y'' = \frac{15}{4}x^{1/2} - \frac{3}{4}x^{-3/2} \\ y''' = \frac{15}{8}x^{-1/2} + \frac{9}{8}x^{-5/2} \end{cases}$

9. $y' = \sec^2 x$, $y'' = 2\sec^2 x \tan x$, $y''' = 4\sec^2 x \tan^2 x + 2\sec^4 x$

11. $y' = -2x\sin(x^2)$, $y'' = -2\sin(x^2) - 4x^2\cos(x^2)$, $y''' = -12x\cos(x^2) + 8x^3\sin(x^2)$

13. $(-1)^n n! x^{-(n+1)}$

15. $n!(2-x)^{-(n+1)}$

17. $(-1)^n n! b^n (a+bx)^{-(n+1)}$

19. $f^{(n)} = \begin{cases} (-1)^k a^n \cos(ax) & \text{if } n = 2k \\ (-1)^{k+1} a^n \sin(ax) & \text{if } n = 2k+1 \end{cases}$ where $k = 0, 1, 2, \ldots$

21. $f^{(n)} = (-1)^k[a^n x\sin(ax) - na^{n-1}\cos(ax)]$ if $n = 2k$, or $(-1)^k[a^n x\cos(ax) + na^{n-1}\sin(ax)]$ if $n = 2k+1$, where $k = 0, 1, 2, \ldots$

23. $-\dfrac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2^n} 3^n (1-3x)^{-(2n-1)/2}$, $(n = 2, 3, \ldots)$

31. If $f^{(n)}$ exists on an interval I and f vanishes at $n+1$ distinct points of I then $f^{(n)}$ vanishes at at least one point of I.

Section 2.9 (page 156)

1. $\dfrac{1-y}{2+x}$

3. $\dfrac{2x+y}{3y^2-x}$

5. $\dfrac{2-2xy^3}{3x^2y^2+1}$

7. $-\dfrac{3x^2+2xy}{x^2+4y}$

9. $2x + 3y = 5$

11. $y = x$

13. $y = 1 - \dfrac{4}{4-\pi}\left(x - \dfrac{\pi}{4}\right)$

15. $y = 2 - x$

17. $\dfrac{2(y-1)}{(1-x)^2}$

19. $\dfrac{(2-6y)(1-3x^2)^2}{(3y^2-2y)^3} - \dfrac{6x}{3y^2-2y}$

21. $-a^2/y^3$

23. 0

25. −26

Section 2.10 (page 162)

1. $5x + C$

3. $\frac{2}{3}x^{3/2} + C$

5. $\frac{1}{4}x^4 + C$

7. $-\cos x + C$

9. $a^2x - \frac{1}{3}x^3 + C$

11. $\frac{4}{3}x^{3/2} + \frac{9}{4}x^{4/3} + C$

13. $\frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2 - x + C$

15. $\frac{1}{2}\sin(2x) + C$

17. $\dfrac{-1}{1+x} + C$

19. $\frac{1}{3}(2x+3)^{3/2} + C$

21. $-\cos(x^2) + C$

23. $\tan x - x + C$

25. $(x + \sin x \cos x)/2 + C$

27. $y = \frac{1}{2}x^2 - 2x + 3$, all x

29. $y = 2x^{3/2} - 15$, $(x > 0)$

31. $y = \dfrac{A}{3}(x^3 - 1) + \dfrac{B}{2}(x^2 - 1) + C(x - 1) + 1$, (all x)

33. $y = \sin x + (3/2)$, (all x)

35. $y = 1 + \tan x$, $-\pi/2 < x < \pi/2$

37. $y = x^2 + 5x - 3$, (all x)

39. $y = \dfrac{x^5}{20} - \dfrac{x^2}{2} + 8$, (all x)

41. $y = 1 + x - \cos x$, (all x)

43. $y = 3x - \dfrac{1}{x}$, $(x > 0)$

45. $y = -\dfrac{7\sqrt{x}}{2} + \dfrac{18}{\sqrt{x}}$, $(x > 0)$

Section 2.11 (page 169)

1. (a) $t > 2$, (b) $t < 2$, (c) all t, (d) no t, (e) $t > 2$, (f) $t < 2$, (g) 2, (h) 0

3. (a) $t < -2/\sqrt{3}$ or $t > 2/\sqrt{3}$, (b) $-2/\sqrt{3} < t < 2/\sqrt{3}$, (c) $t > 0$, (d) $t < 0$, (e) $t > 2/\sqrt{3}$ or $-2/\sqrt{3} < t < 0$, (f) $t < -2/\sqrt{3}$ or $0 < t < 2/\sqrt{3}$, (g) $\pm 12/\sqrt{3}$ at $t = \pm 2/\sqrt{3}$, (h) 12

5. acc = 9.8 m/s^2 downward at all times; max height = 4.9 m; ball strikes ground at 9.8 m/s

7. time 27.8 s; distance 771.6 m

9. $4h$ m, $\sqrt{2}v_0$ m/s

11. 400 ft

13. 0.833 km

15. $v = \begin{cases} 2t & \text{if } 0 < t \le 2 \\ 4 & \text{if } 2 < t < 8 \\ 20 - 2t & \text{if } 8 \le t < 10 \end{cases}$

v is continuous for $0 < t < 10$.

$a = \begin{cases} 2 & \text{if } 0 < t < 2 \\ 0 & \text{if } 2 < t < 8 \\ -2 & \text{if } 8 < t < 10 \end{cases}$

a is continuous except at $t = 2$ and $t = 8$. Maximum velocity 4 is attained for $2 \le t \le 8$.

17. 7 s

19. 448 ft

Review Exercises (page 171)

1. $18x + 6$

3. -1

5. $6\pi x + 12y = 6\sqrt{3} + \pi$

7. $\dfrac{\cos x - 1}{(x - \sin x)^2}$

9. $x^{-3/5}(4 - x^{2/5})^{-7/2}$

11. $-2\theta \sec^2\theta \tan\theta$

13. $20x^{19}$

15. $-\sqrt{3}$

17. $-2xf'(3 - x^2)$

19. $2f'(2x)\sqrt{g(x/2)} + \dfrac{f(2x)\,g'(x/2)}{4\sqrt{g(x/2)}}$

21. $f'(x + (g(x))^2)(1 + 2g(x)g'(x))$

23. $\cos x\, f'(\sin x)\, g(\cos x) - \sin x\, f(\sin x)\, g'(\cos x)$

25. $7x + 10y = 24$

27. $\dfrac{x^3}{3} - \dfrac{1}{x} + C$

29. $2\tan x + 3\sec x + C$

31. $4x^3 + 3x^4 - 7$

33. $I_1 = x \sin x + \cos x + C$, $I_2 = \sin x - x\cos x + C$

35. $y = 3x$

37. points $k\pi$ and $k\pi/(n + 1)$, where k is any integer

39. $(0, 0)$, $(\pm 1/\sqrt{2}, 1/2)$, dist. $= \sqrt{3}/2$ units

41. (a) $k = g/R$

43. 15.3 m

45. 80 ft/s, or about 55 mph

Challenging Problems (page 172)

3. (a) 0, (b) 3/8, (c) 12, (d) -48, (e) 3/7, (f) 21

13. $f(m) = C - (m - B)^2/(4A)$

17. (a) $3b^2 > 8ac$

19. (a) 3 s, (b) $t = 7$ s, (c) $t = 12$ s, (d) about 13.07 m/s^2, (e) 197.5 m, (f) 60.3 m.

Chapter 3
Transcendental Functions

Section 3.1 (page 181)

1. $f^{-1}(x) = x + 1$
$\mathcal{D}(f^{-1}) = \mathcal{R}(f) = \mathcal{R}(f^{-1}) = \mathcal{D}(f) = \mathbb{R}$

3. $f^{-1}(x) = x^2 + 1$, $\mathcal{D}(f^{-1}) = \mathcal{R}(f) = [0, \infty[$,
$\mathcal{R}(f^{-1}) = \mathcal{D}(f) = [1, \infty[$

5. $f^{-1}(x) = x^{1/3}$
$\mathcal{D}(f^{-1}) = \mathcal{R}(f) = \mathcal{R}(f^{-1}) = \mathcal{D}(f) = \mathbb{R}$

7. $f^{-1}(x) = -\sqrt{x}$, $\mathcal{D}(f^{-1}) = \mathcal{R}(f) = [0, \infty[$,
$\mathcal{R}(f^{-1}) = \mathcal{D}(f) =]-\infty, 0]$

9. $f^{-1}(x) = \dfrac{1}{x} - 1$, $\mathcal{D}(f^{-1}) = \mathcal{R}(f) = \{x : x \ne 0\}$,
$\mathcal{R}(f^{-1}) = \mathcal{D}(f) = \{x : x \ne -1\}$

11. $f^{-1}(x) = \dfrac{1 - x}{2 + x}$,
$\mathcal{D}(f^{-1}) = \mathcal{R}(f) = \{x : x \ne -2\}$,
$\mathcal{R}(f^{-1}) = \mathcal{D}(f) = \{x : x \ne -1\}$

13. $g^{-1}(x) = f^{-1}(x + 2)$

15. $k^{-1}(x) = f^{-1}\left(-\dfrac{x}{3}\right)$

17. $p^{-1}(x) = f^{-1}\left(\dfrac{1}{x} - 1\right)$

19. $r^{-1}(x) = \dfrac{1}{4}\left(3 - f^{-1}\left(\dfrac{1 - x}{2}\right)\right)$

21. $f^{-1}(x) = \begin{cases} \sqrt{x - 1} & \text{if } x >= 1 \\ x - 1 & \text{if } x < 1 \end{cases}$

23. $h^{-1}(x) = \begin{cases} \sqrt{x-1} & \text{if } x \ge 1 \\ \sqrt{1-x} & \text{if } x < 1 \end{cases}$

25. $g^{-1}(1) = 2$ **29.** $1/[6(f^{-1}(x))^2]$

31. 2.23362 **33.** $\mathbb{R}$, 1

35. $c = 1$, a, b arbitrary, or $a = b = 0$, $c = -1$.

37. no

Section 3.2 (page 185)

1. $\sqrt{3}$ **3.** x^6

5. 3 **7.** $-2x$

9. x **11.** 1

13. 1 **15.** 2

17. $\log_a(x^4 + 4x^2 + 3)$ **19.** 4.728804...

21. $x = (\log_{10} 5)/(\log_{10}(4/5)) \approx -7.212567$

23. $x = 3^{1/5} = 10^{(\log_{10} 3)/5} \approx 1.24573$

29. 1/2 **31.** 0

33. ∞

Section 3.3 (page 195)

1. $\sqrt{e}$ **3.** x^5

5. $-3x$ **7.** $\ln \dfrac{64}{81}$

9. $\ln(x^2(x-2)^5)$ **11.** $x = \dfrac{\ln 2}{\ln(3/2)}$

13. $x = \dfrac{\ln 5 - 9\ln 2}{2\ln 2}$ **15.** $0 < x < 2$

17. $3 < x < 7/2$ **19.** $5e^{5x}$

21. $(1-2x)e^{-2x}$ **23.** $\dfrac{3}{3x-2}$

25. $\dfrac{e^x}{1+e^x}$ **27.** $\dfrac{e^x - e^{-x}}{2}$

29. e^{x+e^x} **31.** $\dfrac{e^x}{(1+e^x)^2}$

33. $e^x(\sin x - \cos x)$ **35.** $\dfrac{1}{x\ln x}$

37. $2x\ln x$ **39.** $(2\ln 5)5^{2x+1}$

41. $t^x x^t \ln t + t^{x+1}x^{t-1}$ **43.** $\dfrac{b}{(bs+c)\ln a}$

45. $x^{\sqrt{x}}\left(\dfrac{1}{\sqrt{x}}\left(\tfrac{1}{2}\ln x + 1\right)\right)$

47. $\sec x$ **49.** $-\dfrac{1}{\sqrt{x^2+a^2}}$

51. $f^{(n)}(x) = e^{ax}(na^{n-1} + a^n x)$, $n = 1, 2, 3, \dots$

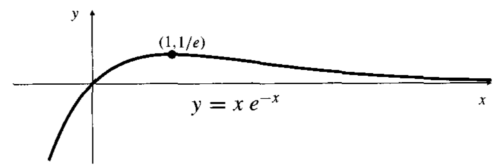
53. $y' = 2xe^{x^2}$, $y'' = 2(1+2x^2)e^{x^2}$,
$y''' = 4(3x+2x^3)e^{x^2}$, $y^{(4)} = 4(3+12x^2+4x^4)e^{x^2}$

55. $f'(x) = x^{x^2+1}(2\ln x + 1)$,
$g'(x) = x^{x^x}x^x\left(\ln x + (\ln x)^2 + \dfrac{1}{x}\right)$;
g grows more rapidly than does f.

57. $f'(x) = f(x)\left(\dfrac{1}{x-1} + \dfrac{1}{x-2} + \dfrac{1}{x-3} + \dfrac{1}{x-4}\right)$

59. $f'(2) = \dfrac{556}{3675}$, $f'(1) = \dfrac{1}{6}$

61. f inc. for $x < 1$, dec. for $x > 1$



63. $y = ex$ **65.** $y = 2e\ln 2(x-1)$

67. $-1/e^2$

69. $f'(x) = (A+B)\cos\ln x + (B-A)\sin\ln x$,
$\int \cos\ln x\,dx = \dfrac{x}{2}(\cos\ln x + \sin\ln x)$,
$\int \sin\ln x\,dx = \dfrac{x}{2}(\sin\ln x - \cos\ln x)$

71. (a) $F_{2B,-2A}(x)$; (b) $-2e^x(\cos x + \sin x)$

Section 3.4 (page 203)

1. 0 **3.** 2

5. 0 **7.** 0

9. 566 **11.** 29.15 years

13. 160.85 years **15.** 4,139 g

17. $7,557.84 **19.** about 14.7 years

21. (a) $f(x) = Ce^{bx} - (a/b)$,
(b) $y = (y_0 + (a/b))e^{bx} - (a/b)$

23. 22.35°C **25.** 6.84 min

29. $(0, -(1/k)\ln(y_0/(y_0 - L)))$, solution $\to -\infty$

31. about 7,671 cases, growing at about 3,028 cases/week

Section 3.5 (page 212)

1. $\pi/3$ **3.** $-\pi/4$

5. 0.7 **7.** $-\pi/3$

9. $\dfrac{\pi}{2} + 0.2$ **11.** $2/\sqrt{5}$

13. $\sqrt{1-x^2}$ **15.** $\dfrac{1}{\sqrt{1+x^2}}$

17. $\dfrac{\sqrt{1-x^2}}{x}$ **19.** $\dfrac{1}{\sqrt{2+x-x^2}}$

21. $\dfrac{-\operatorname{sgn} a}{\sqrt{a^2-(x-b)^2}}$ **23.** $\tan^{-1} t + \dfrac{t}{1+t^2}$

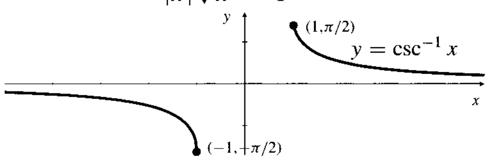
25. $2x\tan^{-1} x + 1$

27. $\dfrac{\sqrt{1-4x^2}\sin^{-1} 2x - 2\sqrt{1-x^2}\sin^{-1} x}{\sqrt{1-x^2}\sqrt{1-4x^2}\left(\sin^{-1} 2x\right)^2}$

29. $\dfrac{x}{\sqrt{(1-x^4)\sin^{-1} x^2}}$ **31.** $\sqrt{\dfrac{a-x}{a+x}}$

33. $\dfrac{\pi-2}{\pi-1}$

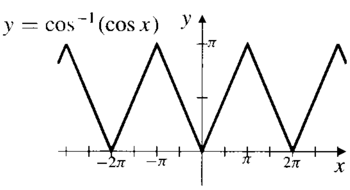
37. $\dfrac{d}{dx}\csc^{-1} x = -\dfrac{1}{|x|\sqrt{x^2-1}}$

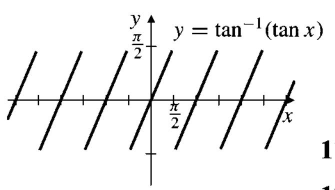


39. $\tan^{-1} x + \cot^{-1} x = -\dfrac{\pi}{2}$ for $x < 0$

41. cont. everywhere, differentiable except at $n\pi$ for integers n

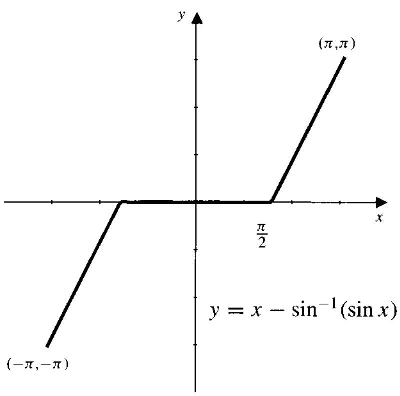
43. continuous and differentiable everywhere except at odd multiples of $\pi/2$





49. $\tan^{-1}\left(\dfrac{x-1}{x+1}\right) - \tan^{-1} x = \dfrac{3\pi}{4}$ on $(-\infty,-1)$

51. $f'(x) = 1 - \operatorname{sgn}(\cos x)$



53. $y = \dfrac{1}{3}\tan^{-1}\dfrac{x}{3} + 2 - \dfrac{\pi}{12}$

55. $y = 4\sin^{-1}\dfrac{x}{5}$

Section 3.6 (page 218)

3. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1+\tanh x \tanh y}$

$\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1-\tanh x \tanh y}$

5. $\dfrac{d}{dx}\sinh^{-1}(x) = \dfrac{1}{\sqrt{x^2+1}}$,

$\dfrac{d}{dx}\cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2-1}}$,

$\dfrac{d}{dx}\tanh^{-1}(x) = \dfrac{1}{1-x^2}$,

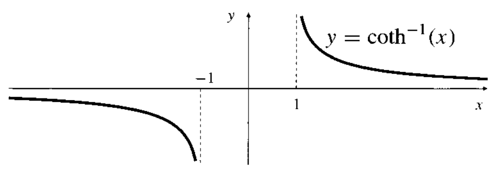
$\displaystyle\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}(x) + C$,

$\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}(x) + C \quad (x>1)$,

$\displaystyle\int \frac{dx}{1-x^2} = \tanh^{-1}(x) + C \quad (-1<x<1)$

7. (a) $\dfrac{x^2-1}{2x}$; (b) $\dfrac{x^2+1}{2x}$; (c) $\dfrac{x^2-1}{x^2+1}$; (d) x^2

9. $\coth^{-1} x = \tanh^{-1}\dfrac{1}{x} = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$, domain: all x such that $|x|>1$, range: all $y \neq 0$, derivative: $-1/(x^2-1)$



11. $f_{A,B} = g_{A+B,A-B}$; $g_{C,D} = f_{(C+D)/2,(C-D)/2}$

13. $y = y_0\cosh k(x-a) + \dfrac{v_0}{k}\sinh k(x-a)$

Section 3.7 (page 228)

1. $y = Ae^{-5t} + Be^{-2t}$ **3.** $y = A + Be^{-2t}$

5. $y = (A+Bt)e^{-4t}$

7. $y = (A\cos t + B\sin t)e^{3t}$

9. $y = (A\cos 2t + B\sin 2t)e^{-t}$

11. $y = (A\cos\sqrt{2}t + B\sin\sqrt{2}t)e^{-t}$

13. $y = \frac{6}{7}e^{t/2} + \frac{1}{7}e^{-3t}$

15. $y = e^{-2t}(2\cos t + 6\sin t)$

25. $y = \frac{3}{10}\sin(10t)$, circ freq 10, freq $\frac{10}{2\pi}$, per $\frac{2\pi}{10}$, amp $\frac{3}{10}$

33. $y = e^{3-t}[2\cos(2(t-3)) + sin(2(t-3))]$

35. $y = -\dfrac{1}{2} + C_1e^t + C_2e^{-2t}$

37. $y = -\dfrac{1}{2}e^{-t} + C_1e^t + C_2e^{-2t}$

39. $y = -e^x \sin x + 3e^x \cos x + C_1 e^x + C_2 e^{-2x}$

41. $y = C_1 t^{r_1} + C_2 t^{r_2}$ **45.** $y = C_1 t^2 + C_2 t^2 \ln t$

Review Exercises (page 230)

1. $1/3$ **3.** both limits are 0

5. max $1/\sqrt{2e}$, min $-1/\sqrt{2e}$

7. $f(x) = 3e^{(x^2/2)-2}$

9. (a) about 13.863%, (b) about 68 days

11. e^{2x} **13.** $y = x$

15. 13.8165% approx.

17. $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x, \cot^{-1} x = \operatorname{sgn} x \sin^{-1}(1/\sqrt{x^2+1})$, $\csc^{-1} x = \sin^{-1}(1/x)$

19. 15°C

Chapter 4
Some Applications of Derivatives

Section 4.1 (page 237)

1. 32 cm²/min

3. increasing at 160π cm²/s

5. (a) $1/(6\pi r)$ km/hr, (b) $1/(6\sqrt{\pi A})$ km/hr

7. $1/(180\pi)$ cm/s **9.** 2 cm²/s

11. increasing at 2 cm³/s **13.** increasing at rate 12

15. increasing at rate $2/\sqrt{5}$

17. $45\sqrt{3}$ km/h **19.** 1/3 m/s, 5/6 m/s

21. 100 tons/day **23.** $16\frac{4}{11}$ min after 3:00

25. $1/(18\pi)$ m/min

27. $9/(6250\pi)$ m/min, 4.64 m

29. 8 m/min **31.** dec. at 126.9 km/h

33. 1/8 units/s **35.** $\sqrt{3}/16$ m/min

37. (a) down at 24/125 m/s, (b) right at 7/125 m/s

39. dec. at 0.0197 rad/s **41.** 0.047 rad/s

Section 4.2 (page 246)

1. abs min 1 at $x = -1$; abs max 3 at $x = 1$

3. abs min 1 at $x = -1$; no max

5. abs min -1 at $x = 0$; abs max 8 at $x = 3$; loc max 3 at $x = -2$

7. abs min $a^3 + a - 4$ at $x = a$; abs max $b^3 + b - 4$ at $x = b$

9. abs max $b^5 + b^3 + 2b$ at $x = b$; no min value

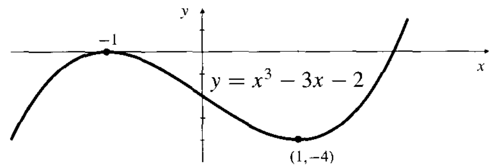
11. no max or min values

13. max 3 at $x = -2$, min 0 at $x = 1$

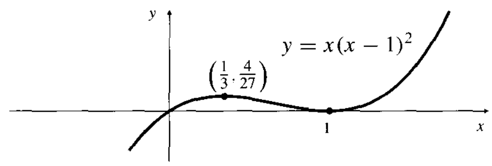
15. abs max 1 at $x = 0$; no min value

17. no max or min value

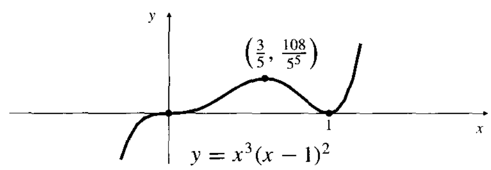
19. loc max at $x = -1$; loc min at $x = 1$



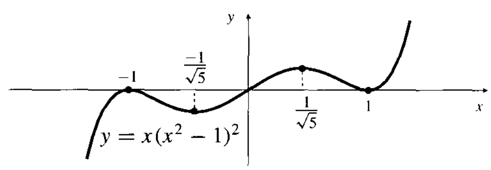
21. loc max at $x = \frac{1}{3}$; loc min at $x = 1$



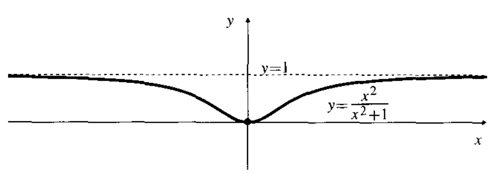
23. loc max at $x = \frac{3}{5}$; loc min at $x = 1$; critical point $x = 0$ is neither max nor min



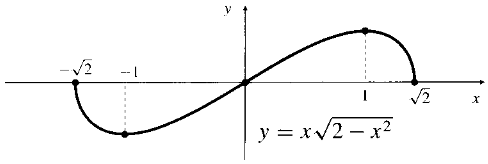
25. loc max at $x = -1$ and $x = 1/\sqrt{5}$; loc min at $x = 1$ and $x = -1/\sqrt{5}$



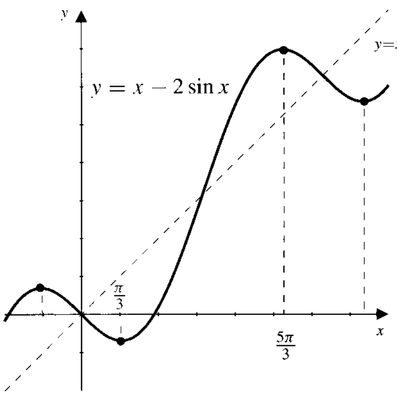
27. abs min at $x = 0$



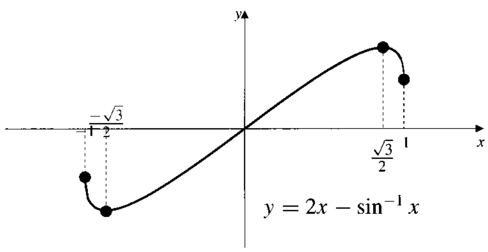
29. loc min at CP $x = -1$ and endpoint SP $x = \sqrt{2}$; loc max at CP $x = 1$ and endpoint SP $x = -\sqrt{2}$



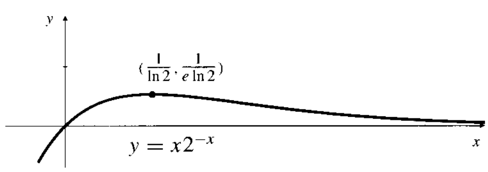
31. loc max at $x = 2n\pi - \dfrac{\pi}{3}$; loc min at $x = 2n\pi + \dfrac{\pi}{3}$ $(n = 0, \pm 1, \pm 2, \ldots)$



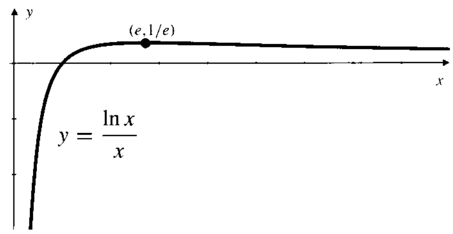
33. loc max at CP $x = \sqrt{3}/2$ and endpoint SP $x = -1$; loc min at CP $x = -\sqrt{3}/2$ and endpoint SP $x = 1$



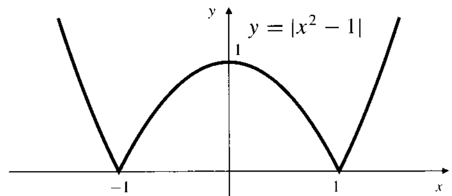
35. abs max at $x = 1/\ln 2$



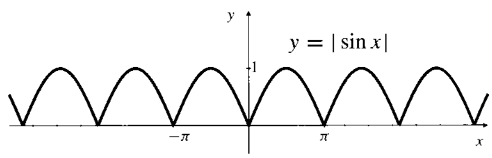
37. abs max at $x = e$



39. loc max at CP $x = 0$; abs min at SPs $x = \pm 1$



41. abs max at CPs $x = (2n+1)\pi/2$; abs min at SPs $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$



43. no max or min

45. max 2, min -2

47. has min, no max

49. yes, no

Section 4.3 (page 251)

1. conc down on $]0, \infty[$

3. conc up on $\mathbb{R}$

5. conc down on $]-1, 0[$ and $]1, \infty[$; conc up on $]-\infty, -1[$ and $]0, 1[$; infl $x = -1, 0, 1$

7. conc down on $]-1, 1[$; conc up on $]-\infty, -1[$ and $]1, \infty[$; infl $x = \pm 1$

9. conc down on $]-2, -2/\sqrt{5}[$ and $]2/\sqrt{5}, 2[$; conc up on $]-\infty, -2[$, $]-2/\sqrt{5}, 2/\sqrt{5}[$ and $]2, \infty[$; infl $x = \pm 2, \ \pm 2/\sqrt{5}$

11. conc down on $]2n\pi, (2n+1)\pi[$; conc up on $](2n-1)\pi, 2n\pi[$, $(n = 0, \pm 1, \pm 2, \ldots)$; infl $x = n\pi$

13. conc down on $\big]n\pi, (n+\frac{1}{2})\pi\big[$; conc up on $\big](n-\frac{1}{2})\pi, n\pi\big[$; infl $x = n\pi/2$, $(n = 0, \pm 1, \pm 2, \ldots)$

15. conc down on $]0, \infty[$, up on $]-\infty, 0[$; infl $x = 0$

17. conc down on $]-1/\sqrt{2}, 1/\sqrt{2}[$, up on $]-\infty, -1/\sqrt{2}[$ and $]1/\sqrt{2}, \infty[$; infl $x = \pm 1/\sqrt{2}$

19. conc down on $]-\infty, -1[$ and $]1, \infty[$; conc up on $]-1, 1[$; infl $x = \pm 1$

21. conc down on $]-\infty, 4[$, up on $]4, \infty[$; infl $x = 4$

23. no concavity, no inflections

25. loc min at $x = 2$; loc max at $x = \frac{2}{3}$

27. loc min at $x = 1/\sqrt[4]{3}$; loc max at $-1/\sqrt[4]{3}$

29. loc max at $x = 1$; loc min at $x = -1$ (both abs)

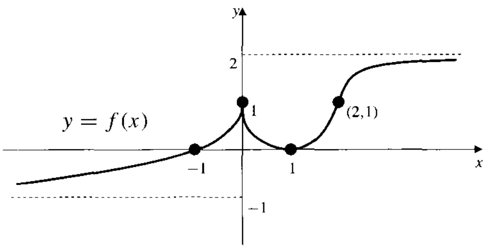
31. loc (and abs) min at $x = 1/e$

Section 4.4 (page 261)

1. (a) g, (b) f'', (c) f, (d) f'

3. (a) $k(x)$, (b) $g(x)$, (c) $f(x)$, (d) $h(x)$

5.



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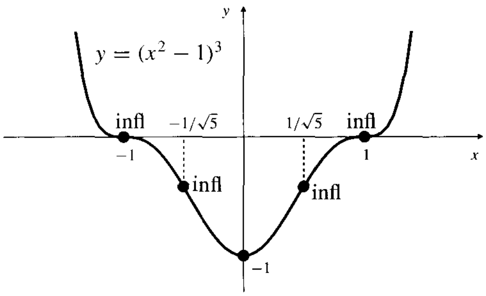

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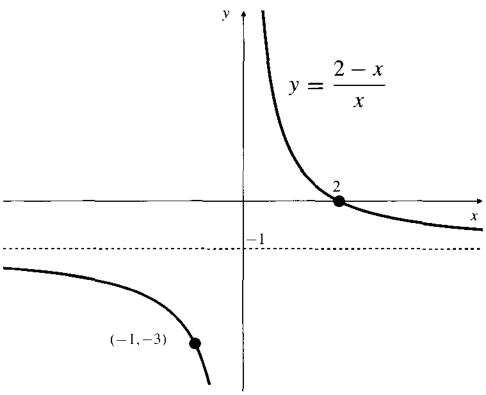

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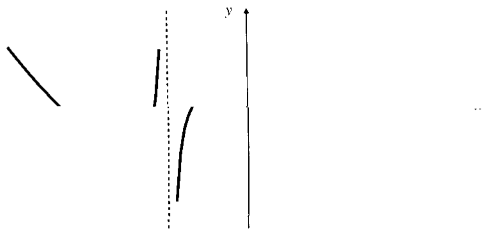

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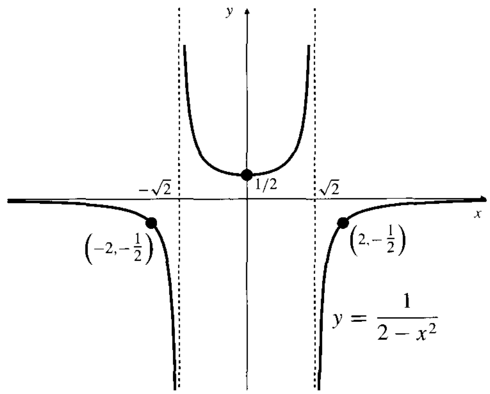

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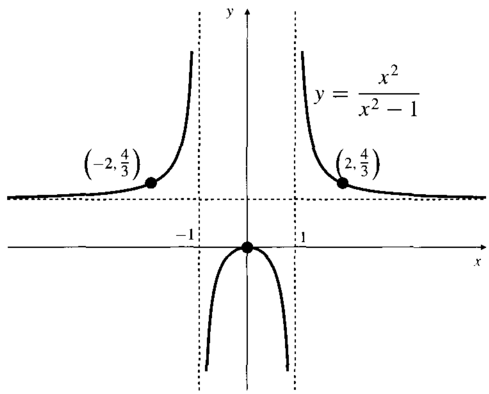

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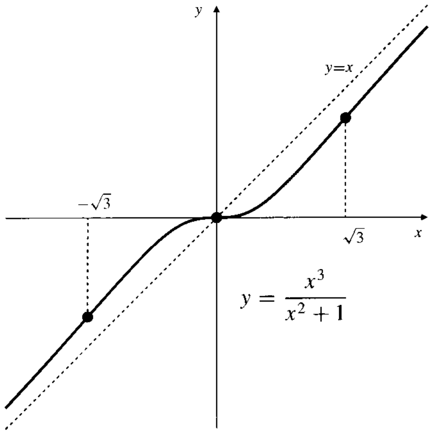
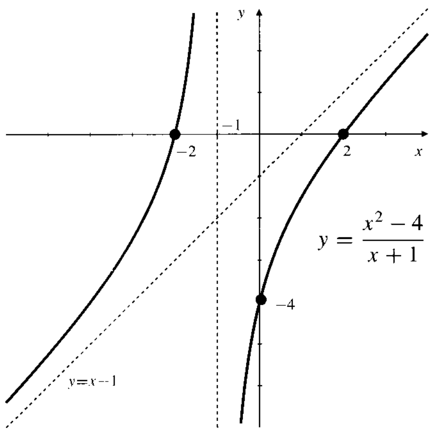


19.



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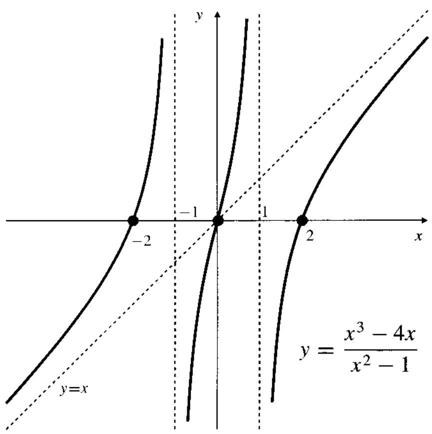

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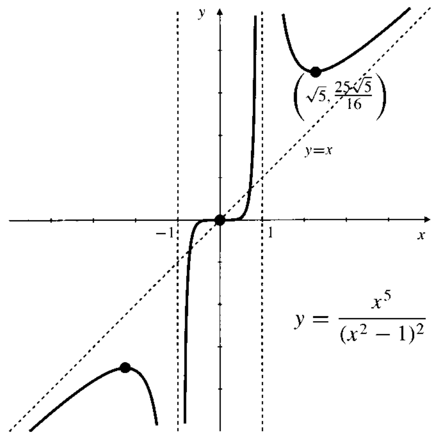

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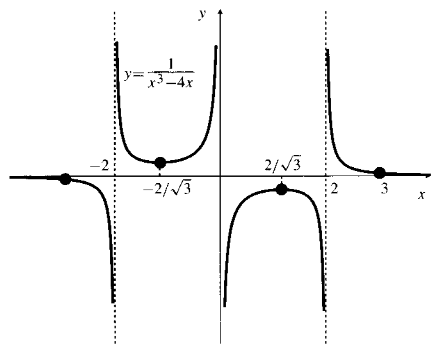

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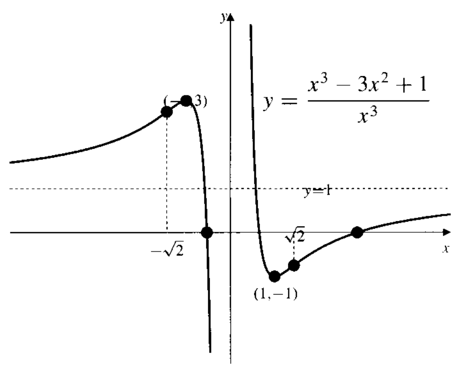

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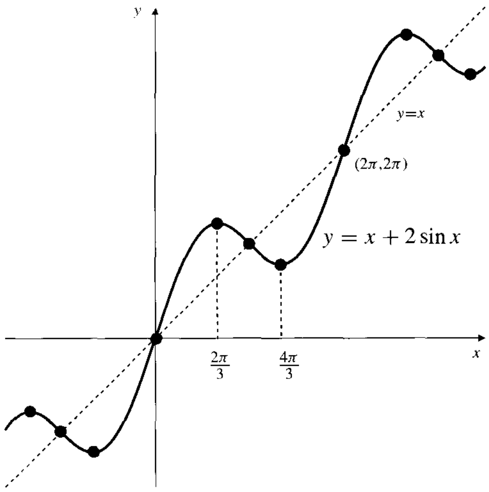

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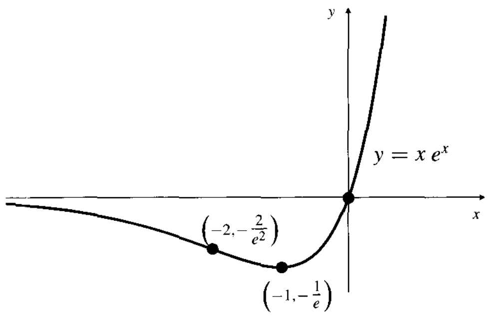

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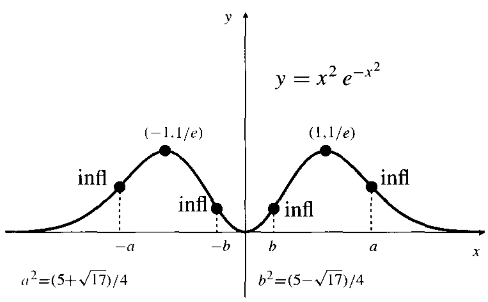

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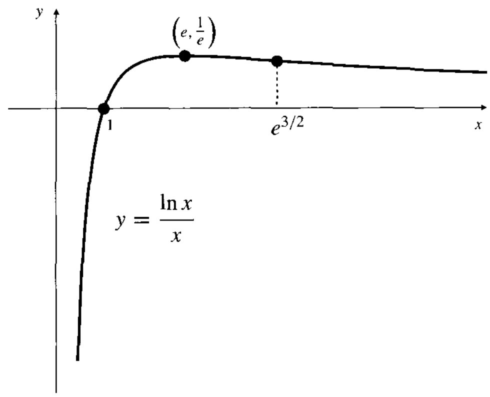

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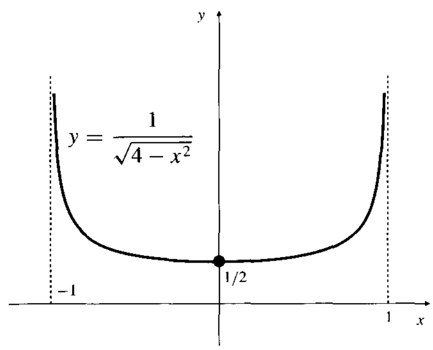

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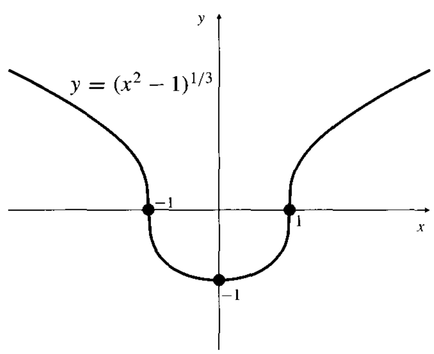


41. $y = 0$. curve crosses asymptote at $x = n\pi$ for every integer n.

Section 4.5 (page 269)

1. $49/4$

3. 20 and 40

5. 71.45

11. R^2 sq. units

13. $2ab$ un^2

15. width $8 + 10\sqrt{2}$ m, height $4 + 5\sqrt{2}$ m

17. rebate \$250

19. point 5 km east of A

21. (a) 0 m, (b) $\pi/(4+\pi)$ m

23. $8\sqrt{3}$ units

25. $\left[(a^{2/3}+b^{2/3})^3+c^2\right]^{1/2}$ units

27. $3^{1/2}/2^{1/3}$ units

29. height $\dfrac{2R}{\sqrt{3}}$, radius $\sqrt{\dfrac{2}{3}}\,R$ units

31. base 2m×2m, height 1m

33. width $\dfrac{20}{4+\pi}$ m, height $\dfrac{10}{4+\pi}$ m

37. width R, depth $\sqrt{3}R$ **39.** $Q = 3L/8$

41. $2\sqrt{6}$ ft **43.** $\dfrac{2\pi}{9\sqrt{3}}R^3$ cubic units

Section 4.6 (page 278)

1. 1.41421356237 **3.** 0.453397651516

5. 1.64809536561, 2.352392647658

7. 0.510973429389

9. infinitely many, 4.49340945791

13. max 1, min −0.11063967219...

15. $x_1 = -a$, $x_2 = a = x_0$. Look for a root half way between x_0 and x_1

17. $x_n = (-1/2)^n \to 0$ (root) as $n \to \infty$.

19. 0.95025 **21.** 0.45340

23. $N(x_n)$ is the Newton's Method approximation x_{n+1}

Section 4.7 (page 284)

1. $6x-9$ **3.** $2-(x/4)$

5. $(7-2x)/27$ **7.** $\pi - x$

9. $(1/4)+(\sqrt{3}/2)(x-(\pi/6))$

11. about 8 cm^2 **13.** about 62.8 mi

15. $\sqrt{50}\approx \frac{99}{14}\approx 7.071429$, error < 0, $|\text{error}| < \frac{1}{2744}\approx 0.0003644$, $]7.07106, 7.071429[$

17. $\sqrt[4]{85}\approx \frac{82}{27}$, error < 0, $|\text{error}| < \frac{1}{2\times 3^6}$, $]3.03635, 3.03704[$

19. $\cos 46^\circ \approx \dfrac{1}{\sqrt{2}}\left(1-\dfrac{\pi}{180}\right)\approx 0.694765$, error < 0, $|\text{error}| < \dfrac{1}{2\sqrt{2}}\left(\dfrac{\pi}{180}\right)^2$, $]0.694658, 0.694765[$

21. $\sin(3.14)\approx \pi - 3.14$, error < 0, $|\text{error}| < (\pi-3.14)^3/2 < 2.02\times 10^{-9}$, $(\pi - 3.14 - (\pi-3.14)^3/2, \pi - 3.14)$

23. $]7.07106, 7.07108[$, $\sqrt{50}\approx 7.07107$

25. $]0.80891, 0.80921[$, $\sqrt[4]{85}\approx 0.80906$

27. $3 \le f(3) \le 13/4$

29. $g(1.8)\approx 0.6$, $|\text{error}| < 0.0208$

31. about 1005 cm^3

Section 4.8 (page 292)

1. $1 - x + \frac{1}{2}x^2 - \dfrac{1}{6}x^3 + \frac{1}{24}x^4$

3. $1+\dfrac{x-e}{e}-\dfrac{(x-e)^2}{2e^2}+\dfrac{(x-e)^3}{3e^3}-\dfrac{(x-e)^4}{4e^4}$

5. $2+\dfrac{x-4}{4}-\dfrac{(x-4)^2}{64}+\dfrac{3(x-4)^3}{1536}$

7. $x^{1/3}\approx 2+\frac{1}{12}(x-8)-\frac{1}{288}(x-8)^2$, $9^{1/3}\approx 2.07986$, $0 <$ error $\le 5/(81\times 256)$, $2.07986 < 9^{1/3} < 2.08010$

9. $\dfrac{1}{x}\approx 1-(x-1)+(x-1)^2$, $\dfrac{1}{1.02}\approx 0.9804$, $-(0.02)^3 \le$ error < 0, $0.980392 \le \dfrac{1}{1.02} < 0.9804$

11. $e^x \approx 1+x+\frac{1}{2}x^2$, $e^{-0.5}\approx 0.625$, $-\frac{1}{6}(0.5)^3 \le$ error < 0, $0.604 \le e^{-0.5} < 0.625$

13. $\sin x = x - \dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+R_7$; $R_7 = \dfrac{\sin X}{8!}x^8$ for some X between 0 and x

15. $\sin x = \dfrac{1}{\sqrt{2}}\left[1+\left(x-\dfrac{\pi}{4}\right)-\dfrac{1}{2!}\left(x-\dfrac{\pi}{4}\right)^2 -\dfrac{1}{3!}\left(x-\dfrac{\pi}{4}\right)^3+\dfrac{1}{4!}\left(x-\dfrac{\pi}{4}\right)^4\right]+R_4$; where $R_4 = \dfrac{\cos X}{5!}\left(x-\dfrac{\pi}{4}\right)^5$ for some X between x and $\pi/4$

17. $\ln x = (x-1)-\dfrac{(x-1)^2}{2}+\dfrac{(x-1)^3}{3}-\dfrac{(x-11)^4}{4}+\dfrac{(x-1)^5}{5}-\dfrac{(x-1)^6}{6}+R_6$; where $R_6 = \dfrac{(x-1)^7}{7X^7}$ for some X between 1 and x

19. $\dfrac{1}{e^3}+\dfrac{3}{e^3}(x+1)+\dfrac{9}{2e^3}(x+1)^2+\dfrac{9}{2e^3}(x+1)^3$

21. $x^2-\frac{1}{3}x^4$ **23.** $1-2x^2+4x^4-8x^6$

25. $x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\cdots+\dfrac{x^{2n+1}}{(2n+1)!}$

27. $e^{-x} = 1 - x + \dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\cdots+(-1)^n\dfrac{x^n}{n!}+R_n$; where $R_n = (-1)^{n+1}\dfrac{e^{-X}x^{n+1}}{(n+1)!}$ for some X between 0 and x; $\dfrac{1}{e}\approx\dfrac{1}{2!}-\dfrac{1}{3!}+\cdots+\dfrac{1}{8!}\approx 0.36788$

29. $1-2x+x^2$ (f is its own best quadratic approximation); (error = 0). $g(x)\approx 4+3x+2x^2$; error $= x^3$; since $g'''(x) = 6 = 3!$, therefore error $= \dfrac{g'''(X)}{3!}x^3$; no improvement possible.

Section 4.9 (page 298)

1. $3/4$ **3.** a/b

5. 1 **7.** 1
9. 0 **11.** $-3/2$
13. 1 **15.** $-1/2$
17. ∞ **19.** $2/\pi$
21. -2 **23.** a
25. 1 **27.** $-1/2$
29. e^{-2} **31.** 0
33. $f''(x)$

Review Exercises (page 299)

1. 6%/min
3. (a) $-1{,}600$ ohms/min, (b) $-1{,}350$ ohms/min
5. 2,000 **7.** $32\pi R^3/81$ un^3
9. 9000 cm^3 **11.** approx 0.057 rad/s
13. about 9.69465 cm **15.** 2.06%
17. $\frac{\pi}{4} + 0.0475 \approx 0.83290$, $|\text{error}| < 0.00011$
19. 0, 1.4055636328
21. approx. $(-1.1462, 0.3178)$

Challenging Problems (page 301)

1. (a) $\dfrac{dx}{dt} = \dfrac{k}{3}(x_0^3 - x^3)$, (b) $V_0/2$
3. (b) 11
5. (c) $y_0(1 - (t/T))^2$, (d) $(1 - (1/\sqrt{2}))T$
7. $P^2(3 - 2\sqrt{2})/4$
9. (a) $\cos^{-1}(r_2/r_1)^2$, (b) $\cos^{-1}(r_2/r_1)^4$.
11. approx 921 cm^3

Chapter 5 Integration

Section 5.1 (page 307)

1. $1^3 + 2^3 + 3^3 + 4^3$ **3.** $3 + 3^2 + 3^3 + \cdots + 3^n$
5. $\dfrac{(-2)^3}{1^2} + \dfrac{(-2)^4}{2^2} + \dfrac{(-2)^5}{3^2} + \cdots + \dfrac{(-2)^n}{(n-2)^2}$
7. $\sin\dfrac{\pi}{3k} + \sin\dfrac{2\pi}{3k} + \sin\dfrac{3\pi}{3k} + \cdots + + \sin\dfrac{k\pi}{3k}$
9. $\sum_{i=5}^{9} i$ **11.** $\sum_{i=2}^{99}(-1)^i i^2$
13. $\sum_{i=0}^{n} x^i$ **15.** $\sum_{i=1}^{n}(-1)^{i-1}/i^2$
17. $\sum_{i=1}^{100}\sin(i-1)$ **19.** 15
21. $n(n+1)(2n+7)/6$ **23.** $\dfrac{\pi(\pi^n - 1)}{\pi - 1} - 3n$
25. $\ln(n!)$ **27.** 400
29. $(x^{2n+1} + 1)/(x + 1)$ **31.** $-4{,}949$
35. $2^m - 1$ **37.** $n/(n+1)$

Section 5.2 (page 314)

1. 3/2 sq. un. **3.** 6 sq. un.
5. 26/3 sq. un. **7.** 15 sq. un.
9. 4 sq. un. **11.** 32/3 sq. un.
13. $3/(2\ln 2)$ sq. un.
15. $\ln(b/a)$, follows from definition of ln
17. 0 **19.** $\pi/4$

Section 5.3 (page 320)

1. $L(f, P_8) = 7/4, \quad U(f, P_8) = 9/4$
3. $L(f, P_4) = \dfrac{e^4 - 1}{e^2(e-1)} \approx 4.22$,
$U(f, P_4) = \dfrac{e^4 - 1}{e(e-1)} \approx 11.48$
5. $L(f, P_6) = \dfrac{\pi}{6}(1 + \sqrt{3}) \approx 1.43$,
$U(f, P_6) = \dfrac{\pi}{6}(3 + \sqrt{3}) \approx 2.48$
7. $L(f, P_n) = \dfrac{n-1}{2n}$, $U(f, P_n) = \dfrac{n+1}{2n}$,
$\int_0^1 x\,dx = \dfrac{1}{2}$
9. $L(f, P_n) = \dfrac{(n-1)^2}{4n^2}$, $U(f, P_n) = \dfrac{(n+1)^2}{4n^2}$,
$\int_0^1 x^3\,dx = \dfrac{1}{4}$
11. $\int_0^1 \sqrt{x}\,dx$ **13.** $\int_0^\pi \sin x\,dx$
15. $\int_0^1 \tan^{-1} x\,dx$

Section 5.4 (page 327)

1. 0 **3.** 8
5. $(b^2 - a^2)/2$ **7.** π
9. 0 **11.** 2π
13. 0 **15.** $(2\pi + 3\sqrt{3})/6$
17. 16 **19.** 32/3
21. $(4 + 3\pi)/12$ **23.** $\ln 2$
25. $\ln 3$ **27.** 4
29. 1 **31.** $\pi/2$
33. 1 **35.** 11/6
37. $\dfrac{\pi}{3} - \sqrt{3}$ **39.** 41/2
41. 3/4 **43.** $k = \overline{f}$

Section 5.5 (page 333)

1. 4
3. 1
5. 9
7. $80\frac{4}{5}$
9. $\dfrac{2-\sqrt{2}}{2\sqrt{2}}$
11. $(1/\sqrt{2})-(1/2)$
13. $e^{\pi}-e^{-\pi}$
15. $(a^e-1)/\ln a$
17. $\pi/2$
19. $\dfrac{\pi}{3}$
21. $\frac{1}{5}$ sq. un.
23. $\frac{32}{3}$ sq. un.
25. $\frac{1}{6}$ sq. un.
27. $\frac{1}{3}$ sq. un.
29. $\frac{1}{12}$ sq. un.
31. 2π sq. un.
33. 3
35. $\frac{16}{3}$
37. $e-1$
39. $\dfrac{\sin x}{x}$
41. $-2\dfrac{\sin x^2}{x}$
43. $\dfrac{\cos t}{1+t^2}$
45. $(\cos x)/(2\sqrt{x})$
47. $f(x)=\pi e^{\pi(x-1)}$
49. $1/x^2$ is not continuous (or even defined) at $x=0$ so the Fundamental Theorem cannot be applied over $[-1,1]$; since $1/x^2>0$ on its domain, we would expect the integral to be positive if it exists at all. (It doesn't.)
51. $F(x)$ has a maximum value at $x=1$ but no minimum value.
53. 2

Section 5.6 (page 341)

1. $-\frac{1}{2}e^{5-2x}+C$
3. $\frac{2}{9}(3x+4)^{3/2}+C$
5. $-\frac{1}{32}(4x^2+1)^{-4}+C$
7. $\frac{1}{2}e^{x^2}+C$
9. $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\sin x\right)+C$
11. $2\ln\left|e^{x/2}-e^{-x/2}\right|+C=\ln\left|e^x-2+e^{-x}\right|+C$
13. $-\frac{2}{5}\sqrt{4-5s}+C$
15. $\frac{1}{2}\sin^{-1}\left(\dfrac{t^2}{2}\right)+C$
17. $-\ln\left(1+e^{-x}\right)+C$
19. $-\frac{1}{2}(\ln\cos x)^2+C$
21. $\frac{1}{2}\tan^{-1}\dfrac{x+3}{2}+C$
23. $\frac{1}{8}\cos^8 x-\frac{1}{6}\cos^6 x+C$
25. $-\dfrac{1}{3a}\cos^3 ax+C$
27. $\frac{5}{16}x-\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C$
29. $\frac{1}{5}\sec^5 x+C$
31. $\frac{2}{3}(\tan x)^{3/2}+\frac{2}{7}(\tan x)^{7/2}+C$
33. $\frac{3}{8}\sin x-\frac{1}{4}\sin(2\sin x)+\frac{1}{32}\sin(4\sin x)+C$
35. $\frac{1}{3}\tan^3 x+C$
37. $-\frac{1}{9}\csc^9 x+\frac{2}{7}\csc^7 x-\frac{1}{5}\csc^5 x+C$
39. $\frac{14}{3}\sqrt{17}+\frac{2}{3}$
41. $3\pi/16$
43. $\ln 2$
45. $2,\ 2(\sqrt{2}-1)$
47. $\pi/32$ sq. un.

Section 5.7 (page 346)

1. $\dfrac{1}{6}$ sq. units
3. $\dfrac{64}{3}$ sq. units
5. $\dfrac{125}{12}$ sq. units
7. $\dfrac{1}{2}$ sq. units
9. $\dfrac{5}{12}$ sq. units
11. $\dfrac{15}{8}-2\ln 2$ sq. units
13. $\dfrac{\pi}{2}-\dfrac{1}{3}$ sq. units
15. $\dfrac{4}{3}$ sq. units
17. $2\sqrt{2}$ sq. units
19. $1-\pi/4$ sq. units
21. $(\pi/8)-\ln\sqrt{2}$ sq. units
23. $(4\pi/3)-2\ln(2+\sqrt{3})$ sq. units
25. $(4/\pi)-1$ sq. units
27. $\dfrac{4}{3}$ sq. units
29. $\dfrac{e}{2}-1$ sq. units

Review Exercises (page 347)

1. sum is $n(n+2)/(n+1)^2$
3. 20/3
5. 4π
7. 0
9. 2
11. $\sin(t^2)$
13. $-4e^{\sin(4s)}$
15. $f(x)=-\frac{1}{2}e^{(3/2)(1-x)}$
17. 9/2 sq. units
19. 3/10 sq. units
21. $(3\sqrt{3}/4)-1$ sq. units
23. $(\frac{1}{6}\sin(2x^3+1)+C$
25. 98/3
27. $(\pi/8)-(1/2)\tan^{-1}(1/2)$
29. $-\cos\sqrt{2s+1}+C$
31. min $-\pi/4$, no max

Chapter 6
Techniques of Integration

Section 6.1 (page 355)

1. $x\sin x+\cos x+C$
3. $\dfrac{1}{\pi}x^2\sin\pi x+\dfrac{2}{\pi^2}x\cos\pi x-\dfrac{2}{\pi^3}\sin\pi x+C$
5. $\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+C$
7. $x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+C$
9. $\left(\frac{1}{2}x^2-\frac{1}{4}\right)\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+C$
11. $\frac{7}{8}\sqrt{2}+\frac{3}{8}\ln(1+\sqrt{2})$

13. $\frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$

15. $\ln(2+\sqrt{3}) - \dfrac{\pi}{6}$

17. $x\tan x - \ln|\sec x| + C$

19. $\dfrac{x}{2}\left[\cos(\ln x) + \sin(\ln x)\right] + C$

21. $\ln x\left(\ln(\ln x) - 1\right) + C$

23. $x\cos^{-1}x - \sqrt{1-x^2} + C$

25. $\dfrac{2\pi}{3} - \ln(2+\sqrt{3})$

27. $\frac{1}{2}(x^2+1)\left(\tan^{-1}x\right)^2 - x\tan^{-1}x + \frac{1}{2}\ln(1+x^2) + C$

29. $\dfrac{1+e^{-\pi}}{2}$ square units

31. $I_n = x(\ln x)^n - nI_{n-1}$,
$I_4 = x\left[(\ln x)^4 - 4(\ln x)^3 + 12(\ln x)^2 - 24(\ln x) + 24\right] + C$

33. $I_n = -\dfrac{1}{n}\sin^{n-1}x\cos x + \dfrac{n-1}{n}I_{n-2}$,
$I_6 = \dfrac{5x}{16} - \cos x\left[\frac{1}{6}\sin^5 x + \frac{5}{24}\sin^3 x + \frac{5}{16}\sin x\right] + C$,
$I_7 = -\cos x\left[\frac{1}{7}\sin^6 x + \frac{6}{35}\sin^4 x + \frac{8}{35}\sin^2 x + \frac{16}{35}\right] + C$

35. $I_n = \dfrac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \dfrac{2n-3}{2a^2(n-1)}I_{n-1}$,
$I_3 = \dfrac{x}{4a^2(x^2+a^2)^2} + \dfrac{3x}{8a^4(x^2+a^2)} + \dfrac{3}{8a^5}\tan^{-1}\dfrac{x}{a} + C$

37. Any conditions that guarantee that $f(b)g'(b) - f'(b)g(b) = f(a)g'(a) - f'(a)g(a)$ will suffice.

Section 6.2 (page 363)

1. $\frac{1}{2}\sin^{-1}(2x) + C$

3. $\frac{9}{2}\sin^{-1}\dfrac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} + C$

5. $-\dfrac{\sqrt{9-x^2}}{9x} + C$

7. $-\sqrt{9-x^2} + \sin^{-1}\dfrac{x}{3} + C$

9. $\frac{1}{3}(9+x^2)^{3/2} - 9\sqrt{9+x^2} + C$

11. $\dfrac{1}{a^2}\dfrac{x}{\sqrt{a^2-x^2}} + C$

13. $\dfrac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\dfrac{x}{a} + C$

15. $\dfrac{1}{2}\sec^{-1}\dfrac{x}{2} + C$

17. $\frac{1}{3}\tan^{-1}\dfrac{x+1}{3} + C$

19. $\frac{1}{32}\tan^{-1}\dfrac{2x+1}{2} + \dfrac{1}{16}\dfrac{2x+1}{4x^2+4x+5} + C$

21. $a\sin^{-1}\dfrac{x-a}{a} - \sqrt{2ax-x^2} + C$

23. $\dfrac{3-x}{4\sqrt{3-2x-x^2}} + C$

25. $\frac{3}{8}\tan^{-1}x + \dfrac{3x^3+5x}{8(1+x^2)^2} + C$

27. $\frac{1}{2}\ln\left(1+\sqrt{1-x^2}\right) - \frac{1}{2}\ln|x| - \dfrac{\sqrt{1-x^2}}{2x^2} + C$

29. $2\sqrt{x} - 4\ln(2+\sqrt{x}) + C$

31. $\frac{6}{7}x^{7/6} - \frac{6}{5}x^{5/6} + \frac{3}{2}x^{2/3} + 2x^{1/2} - 3x^{1/3} - 6x^{1/6} + 3\ln(1+x^{1/3}) + 6\tan^{-1}x^{1/6} + C$

33. $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{8}$

35. $\pi/3$

37. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2\tan(\theta/2)+1}{\sqrt{3}}\right) + C$

39. $\dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{\tan(\theta/2)}{\sqrt{5}}\right) + C$

41. $\dfrac{9}{2\sqrt{2}}\tan^{-1}\dfrac{1}{\sqrt{2}} - \dfrac{1}{2}$ square units

43. $a^2\cos^{-1}\left(\dfrac{b}{a}\right) - b\sqrt{a^2-b^2}$ square units

45. $\dfrac{25}{2}\left(\sin^{-1}\dfrac{4}{5} - \sin^{-1}\dfrac{3}{5}\right) - 12\ln\dfrac{4}{3}$ square units

47. $\dfrac{\ln(Y+\sqrt{1+Y^2})}{2}$ sq. units

Section 6.3 (page 372)

1. $\ln|2x-3| + C$

3. $\dfrac{x}{\pi} - \dfrac{2}{\pi^2}\ln|\pi x + 2| + C$

5. $\dfrac{1}{6}\ln\left|\dfrac{x-3}{x+3}\right| + C$

7. $\dfrac{1}{2a}\ln\left|\dfrac{a+x}{a-x}\right| + C$

9. $x - \frac{4}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$

11. $3\ln|x+1| - 2\ln|x| + C$

13. $\dfrac{1}{3(1-3x)} + C$

15. $-\frac{1}{9}x - \dfrac{13}{54}\ln|2-3x| + \frac{1}{6}\ln|x| + C$

17. $\dfrac{1}{2a^2}\ln\dfrac{|x^2-a^2|}{x^2} + C$

19. $x + \dfrac{a}{3}\ln|x-a| - \dfrac{a}{6}\ln(x^2+ax+a^2) - \dfrac{a}{\sqrt{3}}\tan^{-1}\dfrac{2x+a}{\sqrt{3}a} + C$

21. $\frac{1}{3}\ln|x| - \frac{1}{2}\ln|x-1| + \frac{1}{6}\ln|x-3| + C$

23. $\dfrac{1}{4}\ln\left|\dfrac{x+1}{x-1}\right| - \dfrac{x}{2(x^2-1)} + C$

25. $\dfrac{1}{27}\ln\left|\dfrac{x-3}{x}\right| + \dfrac{1}{9x} + \dfrac{1}{6x^2} + C$

27. $\dfrac{t-1}{4(t^2+1)} - \frac{1}{4}\ln|t+1| + \frac{1}{8}\ln(t^2+1) + C$

29. $\frac{1}{3}\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|+\frac{1}{12}\ln\left(\frac{(2+\sqrt{1-x^2})^2}{3+x^2}\right)+C$

31. $\frac{1}{\sqrt{1+x^2}}+\frac{1}{2}\ln\left|\frac{1-\sqrt{1+x^2}}{1+\sqrt{1+x^2}}\right|+C$

33. $\frac{1-2x^2}{x\sqrt{x^2-1}}+C$

Section 6.4 (page 376)

5. $\frac{x\sqrt{x^2-2}}{2}+\ln|x+\sqrt{x^2-2}|+C$

7. $-\sqrt{3t^2+5}/(5t)+C$

9. $(x^5/3125)(625(\ln x)^4-500(\ln x)^3+300(\ln x)^2 - 120\ln x+24)+C$

11. $(1/6)(2x^2-x-3)\sqrt{2x-x^2}-(1/2)\sin^{-1}(1-x)+C$

13. $(x-2)/(4\sqrt{4x-x^2})+C$

Section 6.5 (page 384)

1. $1/2$
3. $1/2$
5. $3\times 2^{1/3}$
7. $3/2$
9. 3
11. π
13. $1/2$
15. diverges to ∞
17. 2
19. diverges
21. 0
23. 1 sq. units
25. $2\ln 2$ square units
29. 2
31. diverges to ∞
33. converges
35. diverges to ∞
37. diverges to ∞
39. diverges
41. diverges to ∞

Section 6.6 (page 392)

1. $T_4 = 4.75$,
$M_4 = 4.625$,
$T_8 = 4.6875$,
$M_8 = 4.65625$,
$T_{16} = 4.671875$,
Actual errors:
$I - T_4 \approx -0.0833333$,
$I - M_4 \approx 0.0416667$,
$I - T_8 \approx -0.0208333$,
$I - M_8 \approx 0.0104167$,
$I - T_{16} \approx -0.0052083$
Error estimates:
$|I - T_4| \le 0.0833334$,
$|I - M_4| \le 0.0416667$,
$|I - T_8| \le 0.0208334$,
$|I - M_8| \le 0.0104167$,
$|I - T_{16}| \le 0.0052084$

3. $T_4 = 0.9871158$,
$M_4 = 1.0064545$,
$T_8 = 0.9967852$,
$M_8 = 1.0016082$,
$T_{16} = 0.9991967$,
Actual errors:
$I - T_4 \approx 0.0128842$,
$I - M_4 \approx -0.0064545$,
$I - T_8 \approx 0.0032148$,
$I - M_8 \approx -0.0016082$,
$I - T_{16} \approx 0.0008033$
Error estimates:
$|I - T_4| \le 0.020186$,
$|I - M_4| \le 0.010093$,
$|I - T_8| \le 0.005047$,
$|I - M_8| \le 0.002523$,
$|I - T_{16}| \le 0.001262$

5. $T_4 = 46$, $T_8 = 46.7$

7. $T_4 = 3,000\text{ km}^2$, $T_8 = 3,400\text{ km}^2$

9. $T_4 \approx 2.02622$, $M_4 \approx 2.03236$,
$T_8 \approx 2.02929$, $M_8 \approx 2.02982$,
$T_{16} \approx 2.029555$

11. $M_8 \approx 1.3714136$, $T_{16} \approx 1.3704366$, $I \approx 1.371$

Section 6.7 (page 397)

1. $S_4 = S_8 = I$, Errors $= 0$

3. $S_4 \approx 1.0001346$, $S_8 \approx 1.0000083$,
$I - S_4 \approx -0.0001346$, $I - S_8 \approx -0.0000083$

5. 46.93

7. For $f(x) = e^{-x}$:
$|I - S_4| \le 0.000022$, $|I - S_8| \le 0.0000014$;
for $f(x) = \sin x$,
$|I - S_4| \le 0.00021$,
$|I - S_8| \le 0.000013$

9. $S_4 \approx 2.0343333$, $S_8 \approx 2.0303133$,
$S_{16} \approx 2.0296433$

Section 6.8 (page 403)

1. $3\int_0^1 \frac{u\,du}{1+u^3}$

3. $\int_{-\pi/2}^{\pi/2} e^{\sin\theta}\,d\theta$, or $2\int_0^1 \frac{e^{1-u^2}+e^{u^2-1}}{\sqrt{2-u^2}}\,du$

5. $4\int_0^1 \frac{dv}{\sqrt{(2-v^2)(2-2v^2+v^4)}}$

7. $T_2 \approx 0.603553$ $T_4 \approx 0.643283$,
$T_8 \approx 0.658130$, $T_{16} \approx 0.663581$;
Errors: $I - T_2 \approx 0.0631$, $I - T_4 \approx 0.0234$,
$I - T_8 \approx 0.0085$, $I - T_{16} \approx 0.0031$;
Errors do not decrease like $1/n^2$ because the second derivative of $f(x) = \sqrt{x}$ is not bounded on [0, 1].

9. $I \approx 0.74684$ with error less than 10^{-4}; seven terms of the series are needed.

11. $A = 1, u = 1/\sqrt{3}$

13. $A = 5/9, B = 8/9, u = \sqrt{3/5}$

15. $R_1 \approx 0.7471805, \quad R_2 \approx 0.7468337,$ $R_3 \approx 0.7468241, \quad I \approx 0.746824$

17. $R_2 = \frac{2h}{45}\left(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right)$

Review Exercises (Techniques of Integration) (page 404)

1. $\frac{2}{3}\ln|x+2| - \frac{1}{6}\ln|2x+1| + C$

3. $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$ **5.** $\frac{3}{4}\ln\left|\frac{2x-1}{2x+1}\right| + C$

7. $-\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$ **9.** $\frac{1}{5}\left(5x^3-2\right)^{1/3} + C$

11. $\frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{x}{8(4+x^2)} + C$

13. $\frac{1}{2\ln 2}\left(2^x\sqrt{1+4^x} + \ln(2^x + \sqrt{1+4^x})\right) + C$

15. $\frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C$

17. $-e^{-x}\left(\frac{2}{5}\cos 2x + \frac{1}{5}\sin 2x\right) + C$

19. $\frac{x}{10}\left(\cos(3\ln x) + 3\sin(3\ln x)\right) + C$

21. $\frac{1}{4}\left(\ln(1+x^2)\right)^2 + C$

23. $\sin^{-1}\frac{x}{\sqrt{2}} - \frac{x\sqrt{2-x^2}}{2} + C$

25. $\frac{1}{64}\left(\frac{-1}{7(4x+1)^7} + \frac{1}{4(4x+1)^8} - \frac{1}{9(4x+1)^9}\right) + C$

27. $-\frac{1}{4}\cos 4x + \frac{1}{6}\cos^3 4x - \frac{1}{20}\cos^5 4x + C$

29. $-\frac{1}{2}\ln(2e^{-x}+1) + C$

31. $-\frac{1}{2}\sin^2 x - 2\sin x - 4\ln(2-\sin x) + C$

33. $-\frac{\sqrt{1-x^2}}{x} + C$

35. $\frac{1}{48}(1-4x^2)^{3/2} - \frac{1}{16}\sqrt{1-4x^2} + C$

37. $\sqrt{x^2+1} + \ln(x + \sqrt{x^2+1}) + C$

39. $x + \frac{1}{3}\ln|x| + \frac{4}{3}\ln|x-3| - \frac{5}{3}\ln|x+3| + C$

41. $-\frac{1}{10}\cos^{10} x + \frac{1}{6}\cos^{12} x - \frac{1}{14}\cos^{14} x + C$

43. $\frac{1}{2}\ln|x^2+2x-1| - \frac{1}{2\sqrt{2}}\ln\left|\frac{x+1-\sqrt{2}}{x+1+\sqrt{2}}\right| + C$

45. $\frac{1}{3}x^3\sin^{-1}2x + \frac{1}{24}\sqrt{1-4x^2} - \frac{1}{72}(1-4x^2)^{3/2} + C$

47. $\frac{1}{128}\left(3x - \sin(4x) + \frac{1}{8}\sin(8x)\right)$

49. $\tan^{-1}\frac{\sqrt{x}}{2} + C$

51. $\frac{x^2}{2} - 2x + \frac{1}{4}\ln|x| + \frac{1}{2x} + \frac{15}{4}\ln|x+2| + C$

53. $-\frac{1}{2}\cos(2\ln x) + C$ **55.** $\frac{1}{2}\exp\left(2\tan^{-1}x\right) + C$

57. $\frac{1}{4}\left(\ln(3+x^2)\right)^2 + C$ **59.** $\frac{1}{2}\left(\sin^{-1}(x/2)\right)^2 + C$

61. $\sqrt{x^2+6x+10} - 2\ln(x+3+\sqrt{x^2+6x+10}) + C$

63. $\frac{2}{5(2+x^2)^{5/2}} - \frac{1}{3(2+x^2)^{3/2}} + C$

65. $\frac{6}{7}x^{7/6} - \frac{6}{5}x^{5/6} + 2\sqrt{x} - 6x^{1/6} + 6\tan^{-1}x^{1/6} + C$

67. $\frac{2}{3}x^{3/2} - x + 4\sqrt{x} - 4\ln(1+\sqrt{x}) + C$

69. $\frac{1}{2(4-x^2)} + C$

71. $\frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$

73. $\frac{1}{5}\ln\left|\frac{3\tan(x/2)-1}{\tan(x/2)+3}\right| + C$

75. $\frac{1}{2}\ln|\tan(x/2)| - \frac{1}{4}\left(\tan^{-1}(x/2)\right)^2 + C$
$= \frac{1}{4}\left(\ln\left|\frac{1-\cos x}{1+\cos x}\right| - \frac{1-\cos x}{1+\cos x}\right) + C$

77. $2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$

79. $\frac{1}{2}x^2 + \frac{4}{3}\ln|x-2| - \frac{2}{3}\ln(x^2+2x+4)$ $+\frac{4}{\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}} + C$

Review Exercises (Other) (page 405)

1. $I = \frac{1}{2}\left(xe^x\cos x + (x-1)e^x\sin x\right),$
$J = \frac{1}{2}\left((1-x)e^x\cos x + xe^x\sin x\right)$

3. diverges to ∞ **5.** $-4/9$

9. 367,000 m^3

11. $T_8 = 1.61800, \; S_8 = 1.62092, \; I \approx 1.62$

13. (a) $T_4 = 5.526$, $S_4 = 5.504$; (b) $S_8 = 5.504$; (c) yes, because $S_4 = S_8$, and Simpson's Rule is exact for cubics.

Challenging Problems (page 406)

1. (c) $I = \frac{1}{630}, \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$

3. (a) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right),$
(b) $\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x+1) + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x-1)$

7. (a) $a = 7/90, b = 16/45, c = 2/15.$
(b) one interval: approx 0.6321208750, two intervals: approx 0.6321205638, true val: 0.6321205588

Chapter 7
Applications of Integration

Section 7.1 (page 416)

1. $\frac{\pi}{5}$ cu. units
3. $\frac{3\pi}{10}$ cu. units
5. (a) $\frac{16\pi}{15}$ cu. units, (b) $\frac{8\pi}{3}$ cu. units
7. (a) $\frac{27\pi}{2}$ cu. units, (b) $\frac{108\pi}{5}$ cu. units
9. (a) $\frac{15\pi}{4} - \frac{\pi^2}{8}$ cu. units, (b) $\pi(2 - \ln 2)$ cu. units
11. $\frac{10\pi}{3}$ cu. units
13. about 35%
15. $\frac{\pi h}{3}\left(b^2 - 3a^2 + \frac{2a^3}{b}\right)$ cu. units
17. $\frac{\pi}{3}(a-b)^2(2a+b)$ cu. units
19. $\frac{4\pi ab^2}{3}$ cu. units
21. (a) $\pi/2$ cu. units, (b) 2π cu. units
23. $k > 2$
25. about 1, 537 cu. units
27. $8192\pi/105$ cu. units
29. $R = \frac{h\sin\alpha}{\sin\alpha + \cos 2\alpha}$

Section 7.2 (page 420)

1. 6 m^3
3. $\pi/3$ units^3
5. 132 ft^3
7. $\pi a^2 h/2\text{ cm}^3$
9. $3z^2$ sq. units
11. $\frac{16r^3}{3}$ cu. units
13. $72\pi\text{ cm}^3$
15. $\pi r^2(a+b)/2$ cu. units
17. $\frac{16{,}000}{3}$ cu. units
19. $12\pi\sqrt{2}\text{ in}^3$
21. approx 97.28 cm^3

Section 7.3 (page 428)

1. $2\sqrt{5}$ units
3. $52/3$ units
5. $(2/27)(13^{3/2} - 8)$ units
7. 6 units
9. $(e^2+1)/4$ units
11. $\sinh a$ units.
13. $\sqrt{17} + \frac{1}{4}\ln(4+\sqrt{17})$ units
15. $6a$ units
17. 1.0338 units
19. 1.0581
21. $(10^{3/2} - 1)\pi/27$ sq. units
23. $\frac{64\pi}{81}\left[\frac{(13/4)^{5/2} - 1}{5} - \frac{(13/4)^{3/2} - 1}{3}\right]$ sq. units
25. $2\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$ sq. units
27. $2\pi\left(\frac{255}{16} + \ln 4\right)$ sq. units
29. $4\pi^2 ab$ sq. units
31. $8\pi\left(1 + \frac{\ln(2+\sqrt{3})}{2\sqrt{3}}\right)$ sq. units
33. $s = \frac{5}{\pi}\sqrt{4+\pi^2}\,E\left(\frac{\pi}{\sqrt{4+\pi^2}}\right)$
35. $k > -1$
37. (a) π cu. units; (c) "Covering" a surface with paint requires putting on a layer of constant thickness. Far enough to the right, the horn is thinner than any prescribed constant, so it can contain less paint than would be necessary to paint its surface.

Section 7.4 (page 436)

1. mass $\frac{2L}{\pi}$; centre of mass at $\bar{s} = \frac{L}{2}$
3. $m = \frac{1}{4}\pi\delta_0 a^2$; $\bar{x} = \bar{y} = \frac{4a}{3\pi}$
5. $m = \frac{256k}{15}$; $\bar{x} = 0$, $\bar{y} = \frac{16}{7}$
7. $m = \frac{ka^3}{2}$; $\bar{x} = \frac{2a}{3}$, $\bar{y} = \frac{a}{2}$
9. $m = \int_a^b \delta(x)\big(g(x) - f(x)\big)\,dx$;
   $M_{x=0} = \int_a^b x\delta(x)\big(g(x) - f(x)\big)\,dx$, $\bar{x} = M_{x=0}/m$,
   $M_{y=0} = \frac{1}{2}\int_a^b \delta(x)\big((g(x))^2 - (f(x))^2\big)\,dx$,
   $\bar{y} = M_{y=0}/m$
11. Mass is $\frac{8}{3}\pi R^4$ kg. The centre of mass is along the line through the centre of the ball perpendicular to the plane, at a distance $R/10$ m from the centre of the ball on the side opposite the plane.
13. $m = \frac{1}{8}\pi\delta_0 a^4$; $\bar{x} = 16a/(15\pi)$, $\bar{y} = 0$, $\bar{z} = 8a/15$
15. $m = \frac{1}{3}k\pi a^3$; $\bar{x} = 0$, $y = \frac{3a}{2\pi}$
17. about $5.57C/k^{3/2}$

Section 7.5 (page 442)

1. $\left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)$
3. $\left(\frac{\sqrt{2}-1}{\ln(1+\sqrt{2})}, \frac{\pi}{8\ln(1+\sqrt{2})}\right)$
5. $\left(0, \frac{9\sqrt{3} - 4\pi}{4\pi - 3\sqrt{3}}\right)$
7. $\left(\frac{19}{9}, -\frac{1}{3}\right)$

9. The centroid is on the axis of symmetry of the hemisphere half way between the base plane and the vertex.

11. The centroid is on the axis of the cone, one-quarter of the cone's height above the base plane.

13. $(8/9, 11/9)$ **15.** $(0, 2/(3(\pi+2)))$

17. $\left(\frac{\pi}{2}, \frac{\pi}{8}\right)$ **19.** $\left(\frac{2r}{\pi}, \frac{2r}{\pi}\right)$

21. $(1, -2)$ **23.** $\frac{5\pi}{3}$ cu. units

25. $(0.71377, 0.26053)$ **27.** $(1, \frac{1}{5})$

29. $\bar{x} = \frac{M_{x=0}}{A}, \bar{y} = \frac{M_{y=0}}{A}$,

where $A = \int_c^d (g(y) - f(y))\,dy$,

$M_{x=0} = \frac{1}{2}\int_c^d ((g(y))^2 - (f(y))^2)\,dy$,

$M_{y=0} = \int_c^d y(g(y) - f(y))\,dy$

31. diamond orientation, edge upward

Section 7.6 (page 449)

1. (a) 235,200 N, (b) 352,800 N

3. 6.12×10^8 N **5.** 8.92×10^6 N

7. 7.056×10^5 N·m

9. $2450\pi a^3\left(a + \frac{8h}{3}\right)$ N·m

Section 7.7 (page 453)

1. \$11,000 **3.** $\$8(\sqrt{x} - \ln(1+\sqrt{x}))$

5. \$9,063.46 **7.** \$5,865.64

9. \$50,000 **11.** \$11,477.55

13. \$64,872.10 **15.** $\int_0^T e^{-\lambda(t)}P(t)\,dt$

17. about 23,300, \$11,890

Section 7.8 (page 464)

1. (a) $\frac{2}{9}$, (b) $\mu = 2, \sigma^2 = \frac{1}{2}, \sigma = \frac{1}{\sqrt{2}}$,

(c) $\frac{8}{9\sqrt{2}} \approx 0.63$

3. (a) 3, (b) $\mu = \frac{3}{4}, \sigma^2 = \frac{3}{80}, \sigma = \sqrt{\frac{3}{80}}$,

(c) $\frac{69}{20}\sqrt{\frac{3}{80}} \approx 0.668$

5. (a) 6, (b) $\mu = \frac{1}{2}, \sigma^2 = \frac{1}{20}, \sigma = \sqrt{\frac{1}{20}}$,

(c) $\frac{7}{5\sqrt{5}} \approx 0.626$

7. (a) $\frac{2}{\sqrt{\pi}}$, (b) $\mu = \frac{1}{\sqrt{\pi}} \approx 0.0.564, \sigma^2 = \frac{\pi-2}{2\pi}$,

$\sigma = \sqrt{\frac{\pi-2}{2\pi}} \approx 0.426$, (c) Pr$\approx 0.68$

11. (a) 0, (b) $e^{-3} \approx 0.05$, (c) ≈ 0.046

13. approximately 0.006

Section 7.9 (page 472)

1. $y^2 = Cx$ **3.** $x^3 - y^3 = C$

5. $Y = Ce^{t^2/2}$ **7.** $y = \frac{Ce^{2x} - 1}{Ce^{2x} + 1}$

9. $y = -\ln\left(Ce^{-2t} - \frac{1}{2}\right)$ **11.** $y = x^3 + Cx^2$

13. $y = \frac{3}{2} + Ce^{-2x}$ **15.** $y = x - 1 + Ce^{-x}$

17. $y = \sqrt{4 + x^2}$ **19.** $y = \frac{2x}{1+x}$, $(x > 0)$

21. If $a = b$, the given solution is indeterminate 0/0; in this case the solution is $x = a^2kt/(1 + akt)$.

23. $v = \sqrt{\frac{mg}{k}}\,\frac{e^{2\sqrt{kg/mt}} - 1}{e^{2\sqrt{kg/mt}} + 1}$, $v \to \sqrt{\frac{mg}{k}}$

25. the hyperbolas $x^2 - y^2 = C$

Review Exercises (page 473)

1. about 833

3. $a \approx 1.1904, b \approx 0.0476$

5. $a = 2.1773$ **7.** $\left(\frac{8}{3\pi}, \frac{4}{3\pi}\right)$

9. about 27,726 N·cm **11.** $y = 4(x-1)^3$

13. \$8,798.85

Challenging Problems (page 473)

1. (b) $\ln 2/(2\pi)$, (c) $\pi/(4k(k^2+1))$

3. $y = (r/h^3)x^3 - 3(r/h^2)x^2 + 3(r/h)x$

5. $b = -a = 27/2$ **7.** $1/\pi$

9. (a) $S(a, a, c) = 2\pi a^2 + \frac{2\pi ac^2}{\sqrt{a^2 - c^2}}\ln\left(\frac{a + \sqrt{a^2 - c^2}}{c}\right)$.

(b) $S(a, c, c) = 2\pi c^2 + \frac{2\pi a^2 c}{\sqrt{a^2 - c^2}}\cos^{-1}\left(\frac{c}{a}\right)$.

(c) $S(a, b, c) \approx \frac{b - c}{a - c}S(a, a, c) + \frac{a - b}{a - c}S(a, c, c)$.

(d) $S(3, 2, 1) \approx 49.595$.

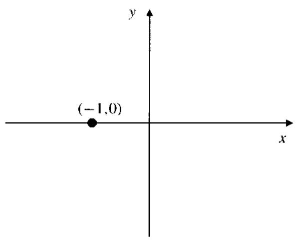
Chapter 8
Conics, Parametric Curves, and Polar Curves

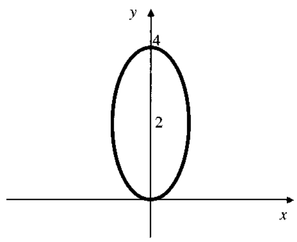
Section 8.1 (page 487)

1. $(x^2/5) + (y^2/9) = 1$ **3.** $(x-2)^2 = 16 - 4y$

5. $3y^2 - x^2 = 3$

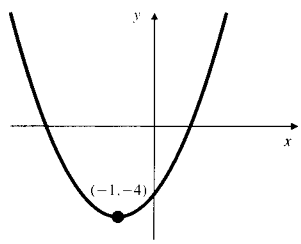
7. single point $(-1, 0)$



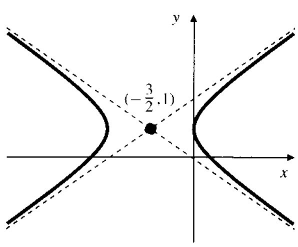


9. ellipse, centre (0,2)

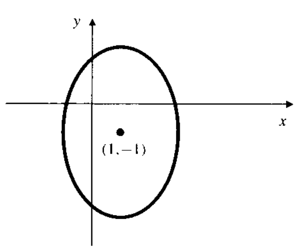
11. parabola, vertex $(-1, -4)$



13. hyperbola, centre $\left(-\frac{3}{2}, 1\right)$
asymptotes
$2x+3 = \pm 2^{3/2}(y-1)$

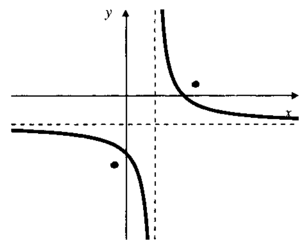


15. ellipse, centre $(1, -1)$

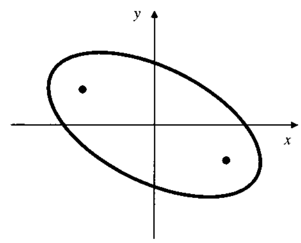


17. $y^2 - 8y = 16x$ or $y^2 - 8y = -4x$

19. rectangular hyperbola, centre $(1, -1)$,
semiaxes $a = b = \sqrt{2}$,
eccentricity $\sqrt{2}$,
foci $(\sqrt{2}+1, \sqrt{2}-1)$,
$(-\sqrt{2}+1, -\sqrt{2}-1)$,
asymptotes $x = 1$, $y = -1$



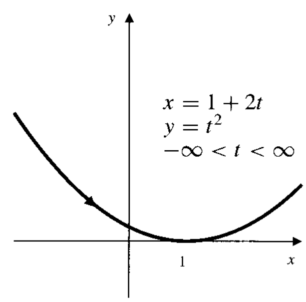
21. ellipse, centre (0,0),
semi-axes $a = 2$, $b = 1$,
foci $\pm\left(2\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right)$



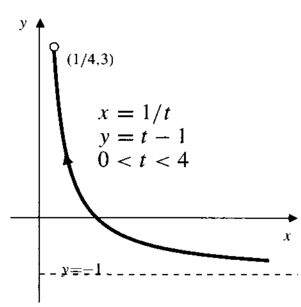
23. $(1-\varepsilon^2)x^2 + y^2 - 2p\varepsilon^2 x = \varepsilon^2 p^2$

Section 8.2 (page 494)

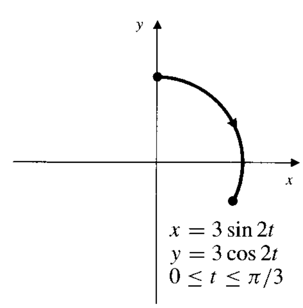
1. $y = (x-1)^2/4$



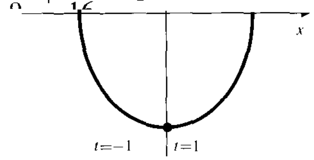
3. $y = (1/x) - 1$



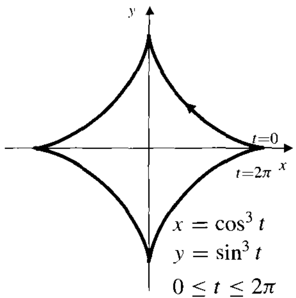
5. $x^2 + y^2 = 9$



7. $\frac{x^2}{9} + \frac{y^2}{16} = 1$



9. $x^{2/3} + y^{2/3} = 1$

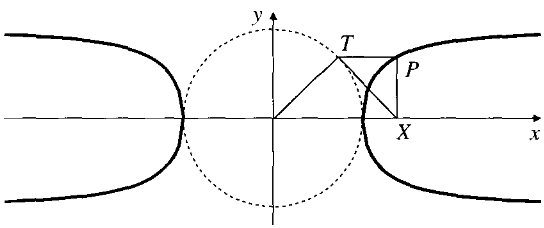


11. the right half of the hyperbola $x^2 - y^2 = 1$

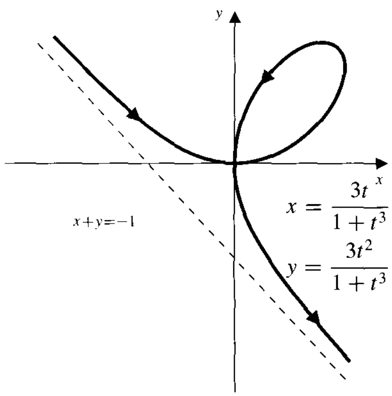
13. the curve starts at the origin and spirals twice counterclockwise around the origin to end at $(4\pi, 0)$

15. $x = m/2, \quad y = m^2/4, \quad (-\infty < m < \infty)$

17. $x = a \sec t,\ y = a \sin t;$
$y^2 = a^2(x^2 - a^2)/x^2$



19. $x^3 + y^3 = 3xy$



Section 8.3 (page 499)

7. horiz. at $(0, \pm 1)$, vert. at $(\pm 1, 1/\sqrt{2})$ and $(\pm 1, -1/\sqrt{2})$

9. $-3/4$

11. $-1/2$

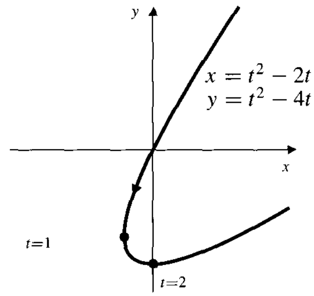
13. $x = t - 2,\ \ y = 4t - 2$

15. slopes ± 1

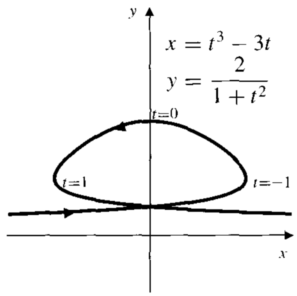
17. not smooth at $t = 0$

19. not smooth at $t = 0$

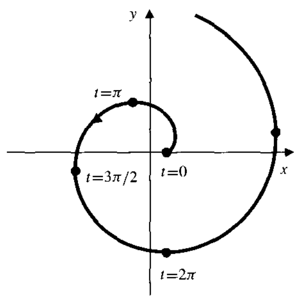
21.



23.



25.



Section 8.4 (page 504)

1. $4\sqrt{2} - 2$ units

3. $6a$ units

5. $\frac{8}{3}\big((1 + \pi^2)^{3/2} - 1\big)$ units

7. 4 units

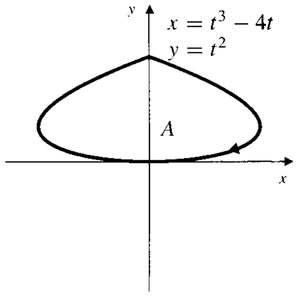
9. $8a$ units

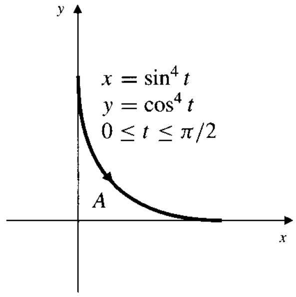
11. $2\sqrt{2}\pi(1 + 2e^{\pi})/5$ sq. units

13. $72\pi(1 + \sqrt{2})/15$ sq. units

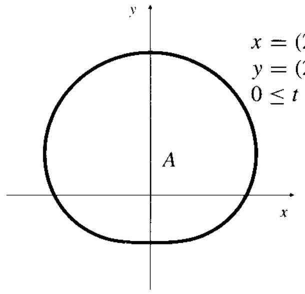
15. 256/15 sq. units

17. 1/6 sq. units





19. $9\pi/2$ sq. units



23. $32\pi a^3/105$ cu. units

Section 8.5 (page 511)

1. $x = 3$, vertical straight line

3. $3y - 4x = 5$, straight line

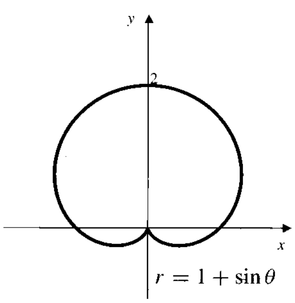
5. $2xy = 1$, rectangular hyperbola

7. $y = x^2 - x$, a parabola

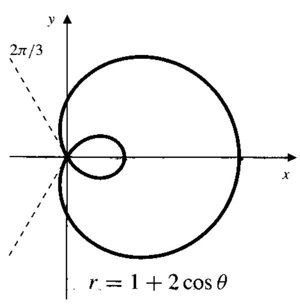
9. $y^2 = 1 + 2x$, a parabola

11. $x^2 - 3y^2 - 8y = 4$, a hyperbola

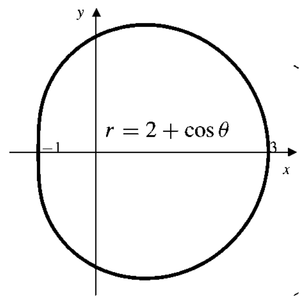
13.



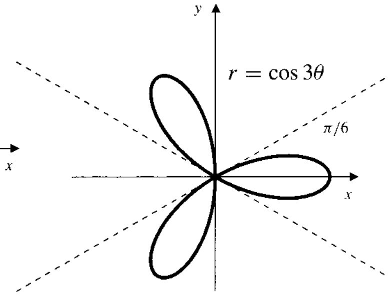
15.



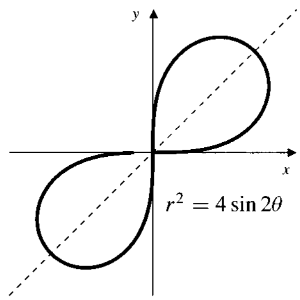
17.



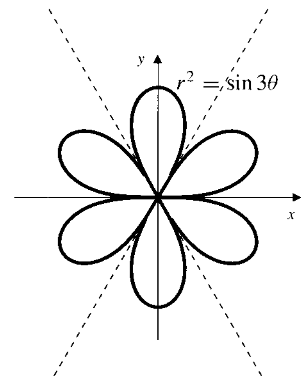
19.



21.



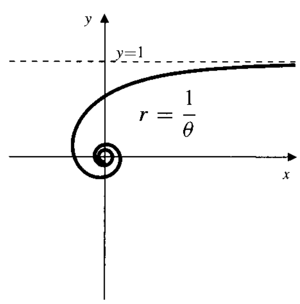
23. $r = \pm\sqrt{\sin 3\theta}$



25. the origin and $[\sqrt{3}/2, \pi/3]$

27. the origin and $[3/2, \pm\pi/3]$

29. asymptote $y = 1$,
$r = 1/(\theta - \alpha)$ has
asymptote$(\cos\alpha)y - (\sin\alpha)x = 1$

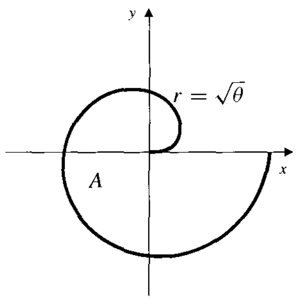


31. $x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$

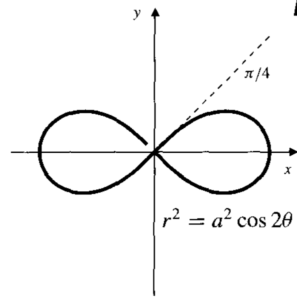
39. $\ln\theta_1 = 1/\theta_1$, point $(-0.108461, 0.556676)$; $\ln\theta_2 = -1/(\theta_2 + \pi)$, point $(-0.182488, -0.178606)$

Section 8.6 (page 515)

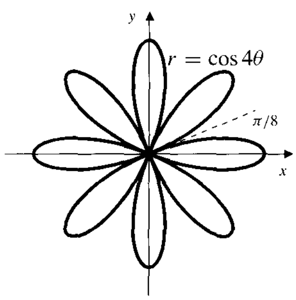
1. π^2 sq. units



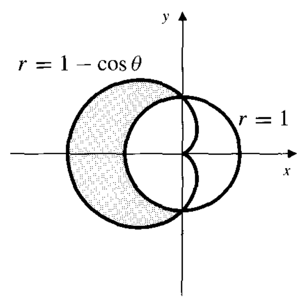
3. a^2 sq. units



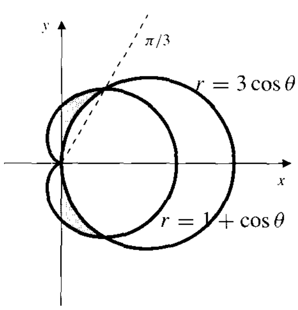
5. $\pi/2$ sq. units



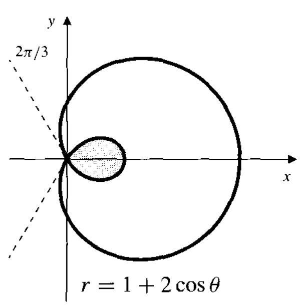
7. $2+(\pi/4)$ sq. units



9. $\pi/4$ sq. units



11. $\pi-\frac{3}{2}\sqrt{3}$ sq. units



13. $\dfrac{\sqrt{1+a^2}}{a}\left(e^{a\pi}-e^{-a\pi}\right)$ units

17. $67.5°$, $-22.5°$

19. $90°$ at (0,0),

$\pm 45°$ at $\left(1-\dfrac{1}{\sqrt{2}}, \dfrac{\pi}{4}\right)$,

$\pm 135°$ at $\left(1+\dfrac{1}{\sqrt{2}}, \dfrac{5\pi}{4}\right)$

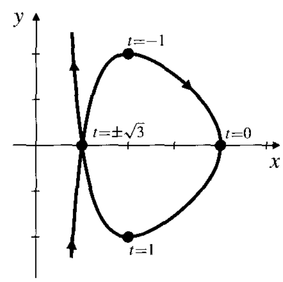
21. horizontal at $\left(\pm\frac{\pi}{4}, \sqrt{2}\right)$, vertical at $(2,0)$ and the origin

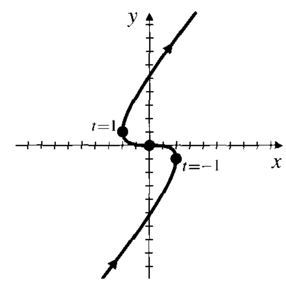
23. horizontal at (0,0), $\left(\frac{2}{3}\sqrt{2}, \pm\tan^{-1}\sqrt{2}\right)$, $\left(\frac{2}{3}\sqrt{2}, \pi\pm\tan^{-1}\sqrt{2}\right)$,
vertical at $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{2}{3}\sqrt{2}, \pm\tan^{-1}(1/\sqrt{2})\right)$, $\left(\frac{2}{3}\sqrt{2}, \pi\pm\tan^{-1}(1/\sqrt{2})\right)$

25. horizontal at $\left(4, -\frac{\pi}{2}\right)$, $\left(1, \frac{\pi}{6}\right)$, $\left(1, \frac{5\pi}{6}\right)$,
vertical at $\left(3, -\frac{\pi}{6}\right)$, $\left(3, -\frac{5\pi}{6}\right)$, no tangent at $\left(0, \frac{\pi}{2}\right)$

Review Exercises (page 516)

1. ellipse, foci $(\pm 1, 0)$, semi-major axis $\sqrt{2}$, semi-minor axis 1
3. parabola, vertex $(4, 1)$, focus $(15/4, 1)$
5. straight line from $(0, 2)$ to $(2, 0)$
7. the parabola $y = x^2 - 1$ left to right
9. first quadrant part of ellipse $16x^2 + y^2 = 16$ from $(1, 0)$ to $(0, 4)$
11. horizontal tangents at $(2, \pm 2)$ (i.e. $t = \pm 1$)
vertical tangent at $(4, 0)$ (i.e. $t = 0$)

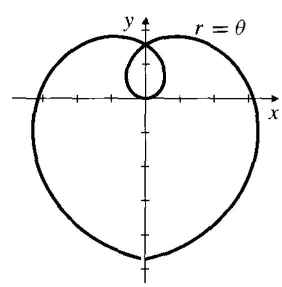


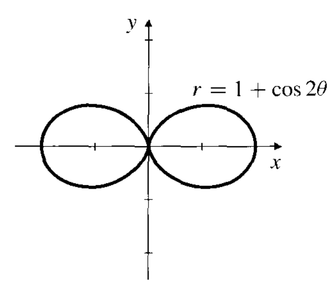


13. horizontal tangent at $(0, 0)$ (i.e. $t = 0$)
vertical tangents at $(2, -1)$ and $(-2, 1)$
(i.e. $t = \pm 1$)

15. 1/2 sq. units

17. $1 + e^2$ units

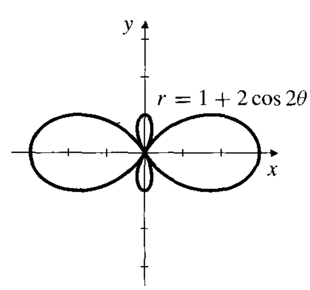




19. $r = \theta$

21. $r = 1 + \cos 2\theta$

23. $r = 1 + 2\cos 2\theta$



25. $\pi + (3\sqrt{3}/4)$ sq. units

27. $(\pi - 3)/2$ sq. units

Challenging Problems (page 516)

1. $16\pi \sec\theta$ cm^2

5. $40\pi/3$ ft^3

7. about 84.65 min

9. $r^2 = \cos(2\theta)$ is the inner curve; area between curves is 1/3 sq. units

Chapter 9
Sequences, Series, and Power Series

Section 9.1 (page 526)

1. bounded, positive, increasing, convergent to 2

3. bounded, positive, convergent to 4

5. bounded below, positive, increasing, divergent to infinity

7. bounded below, positive, increasing, divergent to infinity

9. bounded, positive, decreasing, convergent to 0

11. divergent

13. divergent

15. ∞

17. 0

19. 1

21. e^{-3}

23. 0

25. $1/2$

27. 0

29. 0

31. $\lim_{n\to\infty} a_n = 5$

33. If $\{a_n\}$ is (ultimately) decreasing, then either it is bounded below and therefore convergent, or it is unbounded below and therefore divergent to negative infinity.

Section 9.2 (page 534)

1. $\dfrac{1}{2}$

3. $\dfrac{1}{(2+\pi)^8\big((2+\pi)^2-1\big)}$

5. $\dfrac{25}{4,416}$

7. $\dfrac{8e^4}{e-2}$

9. diverges to ∞

11. $\dfrac{3}{4}$

13. $\dfrac{1}{3}$

15. div. to ∞

17. div. to ∞

19. diverges

21. 14 m

25. If $\{a_n\}$ is ultimately negative, then the series $\sum a_n$ must either converge (if its partial sums are bounded below), or diverge to $-\infty$ (if its partial sums are not bounded below).

27. false, e.g., $\sum \dfrac{(-1)^n}{2^n}$

29. true

31. true

Section 9.3 (page 545)

1. converges

3. diverges to ∞

5. converges

7. diverges to ∞

9. converges

11. diverges to ∞

13. diverges to ∞

15. converges

17. converges

19. diverges to ∞

21. converges

23. converges

25. converges

27. $s_n + \dfrac{1}{3(n+1)^3} \le s \le s_n + \dfrac{1}{3n^3}$; $n = 6$

29. $s_n + \dfrac{2}{\sqrt{n+1}} \le s \le s_n + \dfrac{2}{\sqrt{n}}$; $n = 63$

31. $0 < s - s_n \le \dfrac{n+2}{2^n(n+1)!(2n+3)}$; $n = 4$

33. $0 < s - s_n \le \dfrac{2^n(4n^2+6n+2)}{(2n)!(4n^2+6n)}$; $n = 4$

39. converges, $a_n^{1/n} \to (1/e) < 1$

41. no info from ratio test, but series diverges to infinity since all terms exceed 1.

43. (b) $s \le \dfrac{2}{k(1-k)}$, $k = \frac{1}{2}$,

(c) $0 < s - s_n < \dfrac{(1+k)^{n+1}}{2^n k(1-k)}$, $k = \dfrac{n+2-\sqrt{n^2+8}}{2(n-1)}$ for $n \ge 2$

Section 9.4 (page 553)

1. conv. conditionally

3. conv. conditionally

5. diverges

7. conv. absolutely

9. conv. conditionally

11. diverges

13. 999

15. 13

17. converges absolutely if $-1 < x < 1$, conditionally if $x = -1$, diverges elsewhere

19. converges absolutely if $0 < x < 2$, conditionally if $x = 2$, diverges elsewhere

21. converges absolutely if $-2 < x < 2$, conditionally if $x = -2$, diverges elsewhere

23. converges absolutely if $-\frac{7}{2} < x < \frac{1}{2}$, conditionally if $x = -\frac{7}{2}$, diverges elsewhere

25. AST does not apply directly, but does if we remove all the 0 terms; series converges conditionally

27. (a) false, e.g., $a_n = \dfrac{(-1)^n}{n}$,

(b) false, e.g., $a_n = \dfrac{\sin(n\pi/2)}{n}$ (see Exercise 25),

(c) true

29. converges absolutely for $-1 < x < 1$, conditionally if $x = -1$, diverges elsewhere

Section 9.5 (page 564)

1. centre 0, radius 1, interval $]-1, 1[$

3. centre -2, radius 2, interval $[-4, 0[$

5. centre $\frac{3}{2}$, radius $\frac{1}{2}$, interval $]1, 2[$

7. centre 0, radius ∞, interval $]-\infty, \infty[$

9. $\dfrac{1}{(1-x)^3} = \displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n, \ (-1 < x < 1)$

11. $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=0}^{\infty} (n+1)x^n, \quad (-1 < x < 1)$

13. $\dfrac{1}{(2-x)^2} = \displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n, \quad (-2 < x < 2)$

15. $\ln(2-x) = \ln 2 - \sum_{n=1}^{\infty} \dfrac{x^n}{2^n n}, \quad (-2 \le x < 2)$

17. $\dfrac{1}{x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} (x+2)^n, \quad (-4 < x < 0)$

19. $\dfrac{x^3}{1-2x^2} = \displaystyle\sum_{n=0}^{\infty} 2^n x^{2n+3}, \quad \left(-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}\right)$

21. $\left(-\frac{1}{4}, \frac{1}{4}\right)$; $\quad \dfrac{1}{1+4x}$

23. $[-1, 1)$; $\quad \frac{1}{3}$ if $x = 0$,
$-\dfrac{1}{x^3}\ln(1-x) - \dfrac{1}{x^2} - \dfrac{1}{2x}$ otherwise

25. $(-1, 1)$; $\quad \dfrac{2}{(1-x^2)^2}$

27. $3/4$

29. $\pi^2(\pi+1)/(\pi-1)^3$

31. $\ln(3/2)$

Section 9.6 (page 572)

1. $e^{3x+1} = \sum_{n=0}^{\infty} \dfrac{3^n e}{n!} x^n$, (all x)

3. $\sin\left(x - \dfrac{\pi}{4}\right)$
$= \dfrac{1}{\sqrt{2}} \displaystyle\sum_{n=0}^{\infty} (-1)^n \left[-\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right]$, (all x)

5. $x^2 \sin\left(\dfrac{x}{3}\right) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^{2n+1}(2n+1)!} x^{2n+3}$, (all x)

7. $\sin x \cos x = \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n}}{(2n+1)!} x^{2n+1}$, (all x)

9. $\dfrac{1+x^3}{1+x^2} = 1 - x^2 + \displaystyle\sum_{n=2}^{\infty} (-1)^n \left(x^{2n-1} + x^{2n}\right)$,
$(-1 < x < 1)$

11. $\ln \dfrac{1-x}{1+x} = -2 \displaystyle\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}, \quad (-1 < x < 1)$

13. $\cosh x - \cos x = 2 \sum_{n=0}^{\infty} \dfrac{x^{4n+2}}{(4n+2)!}$, (all x)

15. $e^{-2x} = e^2 \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n}{n!} (x+1)^n$, (all x)

17. $\cos x = \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}}{(2n)!} (x-\pi)^{2n}$, (all x)

19. $\ln 4 + \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{4^n n} (x-2)^n, \quad (-2 < x \le 6)$

21. $\sin x - \cos x =$
$\sqrt{2} \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} \left(x - \dfrac{\pi}{4}\right)^{2n+1}$, (all x)

23. $\dfrac{1}{x^2} = \dfrac{1}{4} \displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2^n} (x+2)^n, \quad (-4 < x < 0)$

25. $(x-1) + \sum_{n=2}^{\infty} \dfrac{(-1)^n}{n(n-1)} (x-1)^n, \quad (0 \le x \le 2)$

27. $1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24}$

29. $x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$

31. $1 + \dfrac{x}{2} - \dfrac{x^2}{8}$

33. e^{x^2} (all x)

35. $\dfrac{e^x - e^{-x}}{2x} = \dfrac{\sinh x}{x}$ if $x \ne 0$, 1 if $x = 0$

37. (a) $1 + x + x^2$, (b) $3 + 3(x-1) + (x-1)^2$

Section 9.7 (page 576)

1. 1.22140

3. 3.32011

5. 0.99619

7. −0.10533

9. 0.42262

11. 1.54306

13. $I(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$, (all x)

15. $K(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(n+1)^2} x^{n+1}, \quad (-1 \le x \le 1)$

17. $M(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)(4n+1)} x^{4n+1}$,
$(-1 \le x \le 1)$

19. 0.946

21. 2

23. $-3/25$

25. 0

Section 9.8 (page 580)

1. $\dfrac{1}{720}(0.2)^7$

3. $\dfrac{1}{120}(0.5)^5$

5. $\dfrac{4\sec^2(0.1)\tan^2(0.1) + 2\sec^4(0.1)}{4!\, 10^4}$

7. $\dfrac{24}{120(1.95)^5(20)^5}$

9. $2^x = \sum_{n=0}^{\infty} \dfrac{(x \ln 2)^n}{n!}$, all x

11. $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$, all x

13. $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ -1 < x < 1$

15. $\dfrac{x}{2+3x} = \sum_{n=1}^{\infty}(-1)^{n-1}3^{n-1}\left(\dfrac{x}{2}\right)^n, \ -\dfrac{2}{3} < x < \dfrac{2}{3}$

17. $\sin x = \dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{c_n}{n!}\left(x - \dfrac{\pi}{6}\right)^n$, (for all x), where $c_n = (-1)^{n/2}$ if n is even, and $c_n = (-1)^{(n-1)/2}\sqrt{3}$ if n is odd

19. $\ln x = \sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{(x-1)^n}{n}, \ 0 < x \le 2$

21. $\dfrac{1}{x} = -\dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{x+2}{2}\right)^n, \ -4 < x < 0$

Section 9.9 (page 584)

1. $\sqrt{1+x}$
$= 1 + \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}\,1\times 3\times 5\times\cdots\times(2n-3)}{2^n n!}x^n$
$|x| < 1$

3. $\sqrt{4+x}$
$= 2 + \dfrac{x}{4} + 2\sum_{n=2}^{\infty}(-1)^{n-1}\dfrac{1\times 3\times 5\times\cdots\times(2n-3)}{2^{3n}n!}x^n,$
$(-4 < x \le 4)$

5. $\sum_{n=0}^{\infty}(n+1)x^n, \quad |x| < 1$

Section 9.10 (page 589)

1. $y = a_0\left(1 + \sum_{k=1}^{\infty}\dfrac{(x-1)^{4k}}{4(k!)(3)(7)\cdots(4k-1)}\right) + a_1\left(x - 1 + \sum_{k=1}^{\infty}\dfrac{(x-1)^{4k+1}}{4(k!)(5)(9)\cdots(4k+1)}\right)$

3. $y = \sum_{n=0}^{\infty}(-1)^n\left[\dfrac{2^n n!}{(2n)!}x^{2n} + \dfrac{1}{2^{n-1}n!}x^{2n+1}\right]$

5. $y = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots$

7. $y_1 = 1 + \sum_{k=1}^{\infty}\dfrac{(-1)^k x^k}{(k!)(2)(5)(8)\cdots(3k-1)},$
$y_2 = x^{1/3}\left(1 + \sum_{k=0}^{\infty}\dfrac{(-1)^k x^k}{(k!)(4)(7)\cdots(3k+1)}\right)$

Review Exercises (page 589)

1. conv. to 0 **3.** div. to ∞
5. $\lim_{n\to\infty} a_n = \sqrt{2}$ **7.** $4\sqrt{2}/(\sqrt{2}-1)$
9. 2 **11.** converges
13. converges **15.** converges
17. conv. abs. **19.** conv. cond.
21. conv. abs. for x in $(-1, 5)$, cond. for $x = -1$, div. elsewhere

23. 1.202
25. $\sum_{n=0}^{\infty} x^n/3^{n+1}, \ |x| < 3$
27. $1 + \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n}/(ne^n), \ -\sqrt{e} < x \le \sqrt{e}$
29. $x + \sum_{n=1}^{\infty}(-1)^n 2^{2n-1}x^{2n+1}/(2n)!$, all x
31. $(1/2) + \sum_{n=1}^{\infty}\dfrac{(-1)^n\,1\times 4\times 7\times\cdots\times(3n-2)x^n}{2\times 24^n n!}$, $-8 < x \le 8$
33. $\sum_{n=0}^{\infty}(-1)^n(x-\pi)^n/\pi^{n+1}, \ 0 < x < 2\pi$
35. $1 + 2x + 3x^2 + \frac{10}{3}x^3$ **37.** $1 - \frac{1}{2}x^2 + \frac{5}{24}x^4$
39. $\begin{cases}\cos\sqrt{x} & \text{if } x \ge 0\\ \cosh\sqrt{|x|} & \text{if } x < 0\end{cases}$ **41.** $\pi^2/(\pi-1)^2$
43. $\ln(e/(e-1))$ **45.** $1/14$
47. 3, 0.49386

Challenging Problems (page 590)

5. (c) 1.645
7. (a) ∞, (c) e^{-x^2}, (d) $f(x) = e^{x^2}\int_0^x e^{-t^2}\,dt$

Chapter 10
Vectors and Coordinate Geometry in 3-Space

Section 10.1 (page 599)

1. 3 units **3.** $\sqrt{6}$ units
5. $|z|$ units; $\sqrt{y^2+z^2}$ units
7. $\cos^{-1}(-4/9) \approx 116.39°$
9. $\sqrt{3}/2$ sq. units **11.** $\sqrt{n-1}$ units
13. the half-space containing the origin and bounded by the plane passing through $(0, -1, 0)$ perpendicular to the y-axis.
15. the vertical plane (parallel to the z-axis) passing through $(1, 0, 0)$ and $(0, 1, 0)$.
17. the sphere of radius 2 centred at $(1, -2, 3)$.
19. the solid circular cylinder of radius 2 with axis along the x-axis.
21. the parabolic cylinder generated by translating the parabola $z = y^2$ in the yz-plane in the direction of the x-axis .
23. the plane through the points $(6, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 2)$.
25. the straight line through $(1, 0, 0)$ and $(1, 1, 1)$.
27. the circle in which the sphere of radius 2 centred at the origin intersects the sphere of radius 2 with centre $(2, 0, 0)$.
29. the ellipse in which the plane $z = x$ intersects the circular cylinder of radius 1 and axis along the z-axis.
31. the part of the solid circular cylinder of radius 1 and axis along the z-axis lying above or on the plane $z = y$.

37. bdry — the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$; interior — points between these spheres; S is closed

39. bdry of S is S, namely the line $x = y = z$; interior is empty; S closed

Section 10.2 (page 609)

1. (a) $3\mathbf{i} - 2\mathbf{j}$, (b) $-3\mathbf{i} + 2\mathbf{j}$, (c) $2\mathbf{i} - 5\mathbf{j}$, (d) $-2\mathbf{i} + 4\mathbf{j}$,(e) $-\mathbf{i} - 2\mathbf{j}$, (f) $4\mathbf{i} + \mathbf{j}$, (g) $-7\mathbf{i} + 20\mathbf{j}$, (h) $2\mathbf{i} - (5/3)\mathbf{j}$

3. (a) $6\mathbf{i} - 10\mathbf{k}$, $8\mathbf{j}$, $-3\mathbf{i} + 20\mathbf{j} + 5\mathbf{k}$
(b) $5\sqrt{2}$, $5\sqrt{2}$
(c) $\frac{3}{5\sqrt{2}}\mathbf{i} \pm \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$ (d) 18
(e) $\cos^{-1}(9/25) \approx 68.9°$ (f) $18/5\sqrt{2}$
(g) $(27/25)\mathbf{i} + (36/25)\mathbf{j} - (9/5)\mathbf{k}$

9. from southwest at $50\sqrt{2}$ km/h.

11. head at angle θ to the east of AC, where
$$\theta = \sin^{-1} \frac{3}{2\sqrt{1+4k^2}}.$$
The trip not possible if $k < \frac{1}{4}\sqrt{5}$. If $k > \frac{1}{4}\sqrt{5}$, there is a second possible heading, $\pi - \theta$, but the trip will take longer.

13. $t = 2$

15. $\cos^{-1}(2/\sqrt{6}) \approx 35.26°$, $90°$

17. $(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$

19. $\lambda = 1/2$, midpoint, $\lambda = 2/3$, 2/3 of way from P_1 to P_2, $\lambda = -1$, P_1 is midway between this point and P_2.

21. plane through point with position vector $(b/|\mathbf{a}|^2)\mathbf{a}$ perpendicular to $\mathbf{a}$.

23. $\mathbf{x} = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$

25. $(|\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u})/\big||\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}\big|$

31. $\mathbf{u} = (\mathbf{w} \bullet \mathbf{a}/|\mathbf{a}|^2)\mathbf{a}$, $\mathbf{v} = \mathbf{w} - \mathbf{u}$

33. $\mathbf{x} = (\mathbf{a} + K\hat{\mathbf{u}})/(2r)$, $\mathbf{y} = (\mathbf{a} - K\hat{\mathbf{u}})/(2s)$, where $K = \sqrt{|\mathbf{a}|^2 - 4rst}$ and $\hat{\mathbf{u}}$ is any unit vector

35. about 12.373 m

37. about 19 m

Section 10.3 (page 618)

1. $5\mathbf{i} + 13\mathbf{j} + 7\mathbf{k}$

3. $\sqrt{6}$ sq. units

5. $\pm\frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

15. 4/3 cubic units

17. $k = -6$

19. $\lambda = \dfrac{\mathbf{x} \bullet (\mathbf{v}\times\mathbf{w})}{\mathbf{u} \bullet (\mathbf{v}\times\mathbf{w})}$, $\mu = \dfrac{\mathbf{x} \bullet (\mathbf{w}\times\mathbf{u})}{\mathbf{u} \bullet (\mathbf{v}\times\mathbf{w})}$, $\nu = \dfrac{\mathbf{x} \bullet (\mathbf{u}\times\mathbf{v})}{\mathbf{u} \bullet (\mathbf{v}\times\mathbf{w})}$

21. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = -2\mathbf{i}+7\mathbf{j}-4\mathbf{k}$, $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \mathbf{i}+9\mathbf{j}+9\mathbf{k}$; the first is in the plane of $\mathbf{v}$ and $\mathbf{w}$, the second is in the plane of $\mathbf{u}$ and $\mathbf{v}$.

(c) $x^2 + y^2 + z^2 = 1$

3. $x - y + 2z = 0$

5. $7x + 5y - z = 12$

7. $x - 5y - 3z = -7$

9. $x + 6y - 5z = 17$

11. $(\mathbf{r}_1 - \mathbf{r}_2) \bullet [(\mathbf{r}_1 - \mathbf{r}_3)\times(\mathbf{r}_1 - \mathbf{r}_4)] = 0$

13. planes passing through the line $x = 0$, $y + z = 1$ (except the plane $y + z = 1$ itself)

15. $\mathbf{r} = (1 + 2t)\mathbf{i} + (2 - 3t)\mathbf{j} + (3 - 4t)\mathbf{k}$, $(-\infty < t < \infty)$
$x = 1+2t, y = 2-3t, z = 3-4t$, $(-\infty < t < \infty)$
$$\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{-4}$$

17. $\mathbf{r} = t(7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k})$; $x = 7t$, $y = -6t$, $z = -5t$; $x/7 = -y/6 = -z/5$

19. $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$;
$x = 1 + t$, $y = 2 + t$, $z = -1 + t$;
$x - 1 = y - 2 = z + 1$

21. $\frac{x-4}{-5} = \frac{y}{3}$, $z = 7$

25. $\mathbf{r}_i \neq \mathbf{r}_j$, $(i, j = 1, \cdots, 4,\ i \neq j)$,
$\mathbf{v} = (\mathbf{r}_1 - \mathbf{r}_2)\times(\mathbf{r}_3 - \mathbf{r}_4) \neq 0$, $(\mathbf{r}_1 - \mathbf{r}_3) \bullet \mathbf{v} = 0$.

27. $7\sqrt{2}/10$ units

29. $18/\sqrt{69}$ units

31. all lines parallel to the xy-plane and passing through (x_0, y_0, z_0).

33. (x, y, z) satisfies the quadratic if either $A_1x + B_1y + C_1z = D_1$ or $A_2x + B_2y + C_2z = D_2$.

Section 10.5 (page 631)

1. ellipsoid centred at the origin with semiaxes 6, 3 and 2 along the x-, y- and z-axes, respectively.

3. sphere with centre $(1, -2, 3)$ and radius $1/\sqrt{2}$.

5. elliptic paraboloid with vertex at the origin, axis along the z-axis, and cross-section $x^2+2y^2 = 1$ in the plane $z = 1$.

7. hyperboloid of two sheets with vertices $(\pm 2, 0, 0)$ and circular cross-sections in planes $x = c$, $(c^2 > 4)$.

9. hyperbolic paraboloid — same as $z = x^2 - y^2$ but rotated 45° about the z-axis (counterclockwise as seen from above).

11. hyperbolic cylinder parallel to the y-axis, intersecting the xz-plane in the hyperbola $(x^2/4) - z^2 = 1$.

13. parabolic cylinder parallel to the y-axis.

15. circular cone with vertex $(2, 3, 1)$, vertical axis, and semi-vertical angle 45°.

17. circle in the plane $x + y + z = 1$ having centre $(1/3, 1/3, 1/3)$ and radius $\sqrt{11/3}$.

19. a parabola in the plane $z = 1 + x$ having vertex at $(-1/2, 0, 1/2)$ and axis along the line $z = 1 + x$, $y = 0$.

21. $\frac{y}{b} - \frac{z}{c} = \lambda\left(1 - \frac{x}{a}\right)$, $\frac{y}{b} + \frac{z}{c} = \frac{1}{\lambda}\left(1 + \frac{x}{a}\right)$;
$\frac{y}{b} - \frac{z}{c} = \mu\left(1 + \frac{x}{a}\right)$, $\frac{y}{b} + \frac{z}{c} = \frac{1}{\mu}\left(1 - \frac{x}{a}\right)$

23. $\mathbf{a} = \mathbf{i} \pm \mathbf{k}$ (or any multiple)

Section 10.6 (page 641)

1. $\begin{pmatrix} 6 & 7 \\ 5 & -3 \\ 1 & 1 \end{pmatrix}$

3. $\begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}$

5. $\mathcal{A}\mathcal{A}^T = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ $\mathcal{A}^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

7. 36

15. $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

17. $x = 1,\ y = 2,\ z = 3$

19. $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2$

21. neg. def.

23. pos. def.

25. indefinite

Section 10.7 (page 648)

1. 2 units

5.
```
sp:=(U,V)->U&.unitv(V)
```

7.
```
ang:=(U,V)->evalf((180/Pi)
    * arccos(unitv(U)&.unitv(V)))
```

9.
```
VolT:=(U,V,W)->(1/6)*abs(U&.(V&x W))
```

11. $(u, v, x, y, z) = (1, 0, -1, 3, 2)$

13. -935

15. $\begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$

Review Exercises (page 648)

1. plane parallel to y-axis through $(3, 0, 0)$ and $(0, 0, 1)$

3. all points on or above the plane through the origin with normal $\mathbf{i} + \mathbf{j} + \mathbf{k}$

5. circular paraboloid with vertex at $(0, 1, 0)$ and axis along the y-axis, opening in the direction of increasing y

7. hyperbolic paraboloid

9. points inside the ellipsoid with vertices at $(\pm 2, 0, 0)$, $(0, \pm 2, 0$, and $(0, 0, \pm 1)$

11. cone with axis along the x-axis, vertex at the origin, and elliptical cross-sections perpendicular to its axis

13. oblique circular cone (elliptic cone). Cross-sections in horizontal planes $z = k$ are circles of radius 1 with centres at $(k, 0, k)$

15. horizontal line through $(0, 0, 3)$ and $(2, -1, 3)$

17. circle of radius 1 centred at $(1, 1, 1)$ in plane normal to $\mathbf{i} + \mathbf{j} + \mathbf{k}$

19. $2x - y + 3z = 0$

21. $2x + 5y + 3z = 2$

23. $7x + 4y - 8z = 6$

25. $\mathbf{r} = (2 + 3t)\mathbf{i} + (1 + t)\mathbf{j} - (1 + 2t)\mathbf{k}$

27. $x = 3t,\ y = -2t,\ z = 4t$

29. $(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1) = \mathbf{0}$

31. $(3/2)\sqrt{34}$ sq. units

33. $\mathcal{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$

35. *pos.def.*

Challenging Problems (page 649)

5. condition: $\mathbf{a} \bullet \mathbf{b} = 0$,
$\mathbf{x} = \dfrac{\mathbf{b} \times \mathbf{a}}{|\mathbf{a}|^2} + t\mathbf{a}$ (for any scalar t)

Chapter 11 Vector Functions and Curves

Section 11.1 (page 657)

1. $\mathbf{v} = \mathbf{j}$, $v = 1$, $\mathbf{a} = \mathbf{0}$, path is the line $x = 1, z = 0$

3. $\mathbf{v} = 2t\mathbf{j} + \mathbf{k}$, $v = \sqrt{4t^2 + 1}$, $\mathbf{a} = 2\mathbf{j}$, path is the parabola $y = z^2$, in the yz-plane

5. $\mathbf{v} = 2t\mathbf{i} - 2t\mathbf{j}$, $v = 2\sqrt{2}t$, $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j}$, path is the straight half-line $x + y = 0,\ z = 1,\ (x \ge 0)$

7. $\mathbf{v} = -a \sin t\mathbf{i} + a \cos t\mathbf{j} + c\mathbf{k}$, $v = \sqrt{a^2 + c^2}$, $\mathbf{a} = -a \cos t\mathbf{i} - a \sin t\mathbf{j}$, path is a circular helix

9. $\mathbf{v} = -3 \sin t\, \mathbf{i} - 4 \sin t\, \mathbf{j} + 5 \cos t\, \mathbf{k}$, $v = 5$, $\mathbf{a} = -\mathbf{r}$, path is the circle of intersection of the plane $4x = 3y$ with the sphere $x^2 + y^2 + z^2 = 25$

11. $\mathbf{a} = \mathbf{v} = \mathbf{r}$, $v = \sqrt{a^2 + b^2 + c^2}\, e^t$, path is the straight line $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

13. $\mathbf{v} = -(e^{-t} \cos e^t + \sin e^t)\mathbf{i}$
$\quad +(-e^{-t} \sin e^t + \cos e^t)\mathbf{j} - e^t\mathbf{k}$
$v = \sqrt{1 + e^{-2t} + e^{2t}}$
$\mathbf{a} = [(e^{-t} - e^t) \cos e^t + \sin e^t]\mathbf{i}$
$\quad +[(e^{-t} - e^t) \sin e^t - \cos e^t]\mathbf{j} - e^t\mathbf{k}$
The path is a spiral lying on the surface $z = -1/\sqrt{x^2 + y^2}$

15. $\mathbf{a} = -3\pi^2\mathbf{i} - 4\pi^2\mathbf{j}$

17. $\sqrt{3/2}(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

19. $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, $\mathbf{a} = -\frac{8}{9}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

29. $\frac{d}{dt}\big(\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\big) = \frac{d\mathbf{u}}{dt}\times(\mathbf{v}\times\mathbf{w})$
$+\mathbf{u}\times\big(\frac{d\mathbf{v}}{dt}\times\mathbf{w}\big) + \mathbf{u}\times\big(\mathbf{v}\times\frac{d\mathbf{w}}{dt}\big)$

31. $\mathbf{u}''' \bullet (\mathbf{u}\times\mathbf{u}')$

33. $\mathbf{r} = \mathbf{r}_0 e^{2t}$, $\mathbf{a} = 4\mathbf{r}_0 e^{2t}$; the path is a straight line through the origin in the direction of $\mathbf{r}_0$

35. $\mathbf{r} = \mathbf{r}_0 + \dfrac{1 - e^{-ct}}{c}\mathbf{v}_0 - \dfrac{g}{c^2}(ct + e^{-ct} - 1)\mathbf{k}$

Section 11.2 (page 666)

1. $\dfrac{e-1}{e}$, $\dfrac{e^2-1}{e^2}$

3. $\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$; the curve is a circle of radius 1 in the plane $z = 1$

5. 4.76° west of south; $\dfrac{\pi^2 R}{72}$ toward the ground, where R is the radius of the earth

7. (a) tangential only, 90° counterclockwise from $\mathbf{v}$.
(b) tangential only, 90° clockwise from $\mathbf{v}$.
(c) normal only

9. 16.0 hours, 52.7°

Section 11.3 (page 673)

1. $x = \sqrt{a^2 - t^2}$, $y = t$, $0 \le t \le a$

3. $x = a\sin\theta$, $y = -a\cos\theta$, $\frac{\pi}{2} \le \theta \le \pi$

5. $\mathbf{r} = -2t\mathbf{i} + t\mathbf{j} + 4t^2\mathbf{k}$

7. $\mathbf{r} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 3(\cos t + \sin t)\mathbf{k}$

9. $\mathbf{r} = (1 + 2\cos t)\mathbf{i} - 2(1 - \sin t)\mathbf{j}$
$+ (9 + 4\cos t - 8\sin t)\mathbf{k}$

11. Choice (b) leads to $\mathbf{r} = \dfrac{t^2-1}{2}\mathbf{i} + t\mathbf{j} + \dfrac{t^2+1}{2}\mathbf{k}$, which represents the whole parabola. Choices (a) and (c) lead to separate parametrizations for the halves $y \ge 0$ and $y \le 0$ of the parabola. For (a) these are $\mathbf{r} = t\mathbf{i} \pm \sqrt{1+2t}\mathbf{j} + (1+t)\mathbf{k}$, $(t \ge -1/2)$

13. $(17\sqrt{17} - 16\sqrt{2})/27$ units

15. $\displaystyle\int_1^T \frac{\sqrt{4a^2t^4 + b^2t^2 + c^2}}{t}\,dt$ units;
$a(T^2 - 1) + c\ln T$ units

17. $\pi\sqrt{2 + 4\pi^2} + \ln(\sqrt{2}\pi + \sqrt{1 + 2\pi^2})$ units

19. $\sqrt{2e^{4\pi} + 1} - \sqrt{3} + \frac{1}{2}\ln\frac{e^{4\pi}+1-\sqrt{2e^{4\pi}+1}}{e^{4\pi}}$
$-\frac{1}{2}\ln(2 - \sqrt{3})$ units

21. straight line segments from (0, 0) to (1, 1), then to (0, 2)

23. $\mathbf{r} = \frac{1}{\sqrt{A^2+B^2+C^2}}(As\mathbf{i} + Bs\mathbf{j} + Cs\mathbf{k})$

25. $\mathbf{r} = a\left(1 - \dfrac{s}{K}\right)^{3/2}\mathbf{i} + a\left(\dfrac{s}{K}\right)^{3/2}\mathbf{j} + b\left(1 - \dfrac{2s}{K}\right)\mathbf{k}$,
$0 \le s \le K$, $K = (\sqrt{9a^2 + 16b^2})/2$

Section 11.4 (page 682)

1. $\hat{\mathbf{T}} = \frac{1}{\sqrt{1+16t^2+81t^4}}(\mathbf{i} - 4t\mathbf{j} + 9t^2\mathbf{k})$

3. $\hat{\mathbf{T}} = \frac{1}{\sqrt{1+\sin^2 t}}(\cos 2t\mathbf{i} + \sin 2t\mathbf{j} - \sin t\mathbf{k})$

Section 11.5 (page 689)

1. 1/2, 27/2

3. $27/(4\sqrt{2})$

5. $\hat{\mathbf{T}} = (\mathbf{i} + 2\mathbf{j})/\sqrt{5}$, $\hat{\mathbf{N}} = (-2\mathbf{i} + \mathbf{j})/\sqrt{5}$, $\hat{\mathbf{B}} = \mathbf{k}$

7. $\hat{\mathbf{T}} = \dfrac{1}{\sqrt{1 + t^2 + t^4}}(\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})$,
$\hat{\mathbf{B}} = \dfrac{1}{\sqrt{t^4 + 4t^2 + 1}}(t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k})$,
$\hat{\mathbf{N}} = \dfrac{-(t + 2t^3)\mathbf{i} + (1 - t^4)\mathbf{j} + (t^3 + 2t)\mathbf{k}}{\sqrt{t^4 + 4t^2 + 1}\sqrt{1 + t^2 + t^4}}$,
$\kappa = \dfrac{\sqrt{t^4 + 4t^2 + 1}}{(t^4 + t^2 + 1)^{3/2}}$, $\tau = \dfrac{2}{t^4 + 4t^2 + 1}$

9. $\kappa(t) = 1/\sqrt{2}$, $\tau(t) = 0$, curve is a circle in the plane $y + z = 4$, having centre (2, 1, 3) and radius $\sqrt{2}$

11. i) $\hat{\mathbf{T}} = \mathbf{i}$, $\hat{\mathbf{N}} = \dfrac{2\mathbf{j} - \mathbf{k}}{\sqrt{5}}$,
$\hat{\mathbf{B}} = \dfrac{\mathbf{j} + 2\mathbf{k}}{\sqrt{5}}$, $\kappa = \sqrt{5}$, $\tau = 0$
ii) $\hat{\mathbf{T}} = \sqrt{\frac{2}{3}}(\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k})$, $\hat{\mathbf{B}} = \frac{1}{\sqrt{13}}(-\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k})$,
$\hat{\mathbf{N}} = -\frac{1}{\sqrt{39}}(6\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k})$, $\kappa = \frac{2\sqrt{39}}{9}$, $\tau = -\frac{6\sqrt{2}}{13}$

13. max a/b^2, min b/a^2

15. $\kappa = \dfrac{e^x}{(1 + e^{2x})^{3/2}}$,
$\mathbf{r} = (x - 1 - e^{2x})\mathbf{i} + (2e^x + e^{-x})\mathbf{j}$

17. $\frac{3}{2\sqrt{2ar}}$

21. $\mathbf{r} = -4x^3\mathbf{i} + (3x^2 + \frac{1}{2})\mathbf{j}$

23. $f(x) = \frac{1}{8}(15x - 10x^3 + 3x^5)$

Section 11.6 (page 699)

3. velocity: $1/\sqrt{2}$, $1/\sqrt{2}$;
acceleration: $-e^{-\theta}/2$, $e^{-\theta}/2$.

5. $|a_r| = \dfrac{v_0^2}{5}\left(\dfrac{2}{r^2} + \dfrac{1}{r^3}\right)$

7. 42,777 km, the equatorial plane

9. $\frac{T}{4\sqrt{2}}$

13. 3/4

15. $(1/2) - (\epsilon/\pi)$

19. $r = A\sec\omega(\theta - \theta_0)$, $\omega^2 = 1 - (k/h^2)$ if $k < h^2$,
$r = 1/(A + B\theta)$ if $k = h^2$,
$r = Ae^{\omega\theta} + Be^{-\omega\theta}$, $\omega^2 = (k/h^2) - 1$, if $k > h^2$;
there are no bounded orbits that do not approach the origin except in the case $k = h^2$ if $B = 0$ when there are circular orbits. (Now aren't you glad gravitation is an inverse square rather than an inverse cube attraction?)

21. centre $\left(\dfrac{\ell\epsilon}{\epsilon^2 - 1}, 0\right)$;
asymptotes in directions $\theta = \pm\cos^{-1}\left(-\dfrac{1}{\epsilon}\right)$;
semi-transverse axis $a = \dfrac{\ell}{\epsilon^2 - 1}$;
semi-conjugate axis $b = \dfrac{\ell}{\sqrt{\epsilon^2 - 1}}$;
semi-focal separation $c = \dfrac{\ell\epsilon}{\epsilon^2 - 1}$.

Review Exercises (page 701)

3. $\mathbf{v} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$, $\mathbf{a} = (8/3)(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

5. $\kappa = \tau = \sqrt{2}/(e^t + e^{-t})^2$

9. $4a(1 - \cos(T/2))$ units

11. $\mathbf{r}_C(t) = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}$

13. $\hat{\boldsymbol{\rho}} = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}$
$\hat{\boldsymbol{\phi}} = \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k}$
$\hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$
right-handed

Challenging Problems (page 702)

1. (a) $\boldsymbol{\Omega} = \Omega\dfrac{\mathbf{j} + \mathbf{k}}{\sqrt{2}}$, $\Omega \approx 7.272 \times 10^{-5}$.
(b) $\mathbf{a}_C = -\sqrt{2}\Omega v\mathbf{i}$.
(c) about 15.5 cm west of P.

3.
(c) $\mathbf{v}(t) = (\mathbf{v}_0 - (\mathbf{v}_0 \bullet \mathbf{k})\mathbf{k})\cos(\omega t) + (\mathbf{v}_0 \times \mathbf{k})\sin(\omega t) + (\mathbf{v}_0 \bullet \mathbf{k})\mathbf{k}$.
(d) Straight line if $\mathbf{v}_0$ is parallel to $\mathbf{k}$, circle if $\mathbf{v}_0$ is perpendicular to $\mathbf{k}$.

5. (a) $y = (48 + 24x^2 - x^4)/64$

7. (a) Yes, time $\pi a/(v\sqrt{2})$, (b) $\phi = \dfrac{\pi}{2} - \dfrac{vt}{a\sqrt{2}}$,
$\theta = \ln\left[\sec\left(\dfrac{vt}{a\sqrt{2}}\right) + \tan\left(\dfrac{vt}{a\sqrt{2}}\right)\right]$.
(c) infinitely often

Chapter 12
Partial Differentiation

Section 12.1 (page 712)

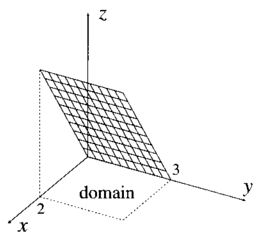
1. all (x, y) with $x \neq y$ **3.** all (x, y) except $(0, 0)$

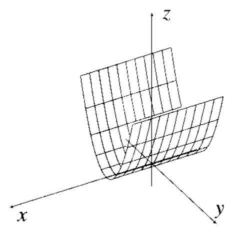
5. all (x, y) satisfying $4x^2 + 9y^2 \geq 36$

7. all (x, y) with $xy > -1$

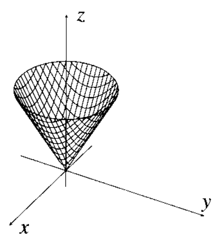
9. all (x, y, z) except $(0, 0, 0)$

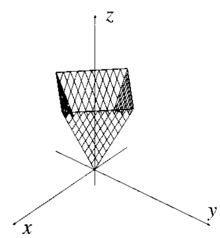
11. $z = f(x, y) = x$ **13.** $z = f(x, y) = y^2$



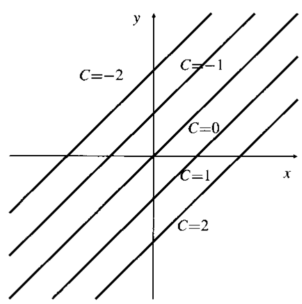


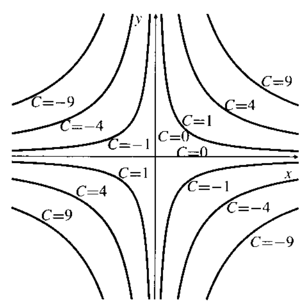
15. $f(x, y) = \sqrt{x^2 + y^2}$ **17.** $f(x, y) = |x| + |y|$



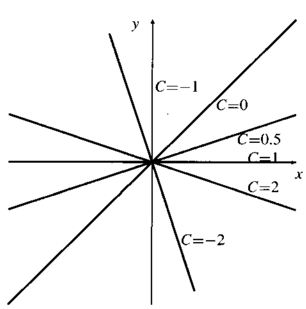


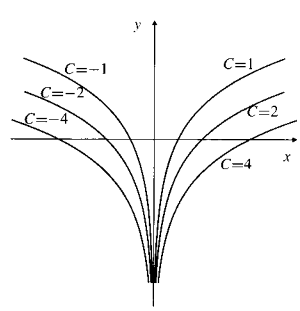
19. $f(x, y) = x - y = C$ **21.** $f(x, y) = xy = C$





23. $f(x, y) = \dfrac{x - y}{x + y} = C$ **25.** $f(x, y) = xe^{-y} = C$





27. At B, because the contours are closer together there

29. a plane containing the y-axis, sloping uphill in the x direction

31. a right-circular cone with base in the xy-plane and vertex at height 5 on the z-axis

33. No, different curves of the family must not intersect in the region.

35. (a) $\sqrt{x^2+y^2}$, (b) $(x^2+y^2)^{1/4}$, (c) x^2+y^2, (d) $e^{\sqrt{x^2+y^2}}$

37. spheres centred at the origin

39. circular cylinders with axis along the z-axis

41. regular octahedra with vertices on the coordinate axes

Section 12.2 (page 717)

1. 2

3. does not exist

5. -1

7. 0

9. does not exist

11. 0

13. $f(0,0)=1$

15. all (x,y) such that $x\neq\pm y$; yes; yes $f(x,x)=\frac{1}{2x}$ makes f continuous at (x,x) for $x\neq 0$; no, f has no continuous extension to the line $x+y=0$.

17. no, yes

19. $a=c=0$, $b\neq 0$

23. a surface having no tears in it, meeting vertical lines through points of the region exactly once

Section 12.3 (page 724)

1. $f_1(x,y)=f_1(3,2)=1$, $f_2(x,y)=f_2(3,2)=-1$

3. $f_1=3x^2y^4z^5$, $f_2=4x^3y^3z^5$, $f_3=5x^3y^4z^4$
All three vanish at $(0,-1,-1)$.

5. $\dfrac{\partial z}{\partial x}=\dfrac{-y}{x^2+y^2}$, $\dfrac{\partial z}{\partial y}=\dfrac{x}{x^2+y^2}$,
At $(-1,1)$: $\dfrac{\partial z}{\partial x}=-\dfrac{1}{2}$, $\dfrac{\partial z}{\partial y}=-\dfrac{1}{2}$

7. $f_1=\sqrt{y}\cos(x\sqrt{y})$, $f_2=\dfrac{x\cos(x\sqrt{y})}{2\sqrt{y}}$,
At $(\pi/3,4)$: $f_1=-1$, $f_2=-\pi/24$

9. $\dfrac{\partial w}{\partial x}=y\ln z\, x^{(y\ln z-1)}$, $\dfrac{\partial w}{\partial y}=\ln x\,\ln z\, x^{y\ln z}$,
$\dfrac{\partial w}{\partial z}=\dfrac{y\ln x}{z}x^{y\ln z}$
At $(e,2,e)$: $\dfrac{\partial w}{\partial x}=\dfrac{\partial w}{\partial z}=2e$, $\dfrac{\partial w}{\partial y}=e^2$.

11. $f_1(0,0=2$, $f_2(0,0)=-1/3$

13. $z=-4x-2y-3$; $\frac{x+2}{-4}=\frac{y-1}{-2}=\frac{z-3}{-1}$

15. $z=\dfrac{1}{\sqrt{2}}\left(1-\dfrac{x-\pi}{4}+\dfrac{\pi}{16}(y-4)\right)$;
$\dfrac{x-\pi}{-1/4\sqrt{2}}=\dfrac{y-4}{\pi/16\sqrt{2}}=\dfrac{z-1/\sqrt{2}}{-1}$

17. $z=\frac{2}{5}+\frac{3x}{25}-\frac{4y}{25}$; $\frac{x-1}{3}=\frac{y-2}{-4}=\frac{z-1/5}{-25}$

19. $z=\ln 5+\frac{2}{5}(x-1)-\frac{4}{5}(y+2)$;
$\dfrac{x-1}{2/5}=\dfrac{y+2}{-4/5}=\dfrac{z-\ln 5}{-1}$

21. $z=\dfrac{x+y}{2}-\dfrac{\pi}{4}$; $2(x-1)=2(y+1)=-z-\dfrac{\pi}{4}$

23. $(0,0)$, $(1,1)$, $(-1,-1)$

33. $w=f(a,b,c)+f_1(a,b,c)(x-a)+f_2(a,b,c)(y-b)+f_3(a,b,c)(z-c)$

35. $\sqrt{7}/4$ units

37. $f_1(0,0)=1$, $f_2(0,0)$ does not exist.

39. f is continuous at $(0,0)$; f_1 and f_2 are not.

Section 12.4 (page 731)

1. $\dfrac{\partial^2 z}{\partial x^2}=2(1+y^2)$, $\dfrac{\partial^2 z}{\partial x\partial y}=4xy$, $\dfrac{\partial^2 z}{\partial y^2}=2x^2$

3. $\dfrac{\partial^2 w}{\partial x^2}=6xy^3z^3$, $\dfrac{\partial^2 w}{\partial y^2}=6x^3yz^3$,
$\dfrac{\partial^2 w}{\partial z^2}=6x^3y^3z$, $\dfrac{\partial^2 w}{\partial x\partial y}=9x^2y^2z^3$,
$\dfrac{\partial^2 w}{\partial x\partial z}=9x^2y^3z^2$, $\dfrac{\partial^2 w}{\partial y\partial z}=9x^3y^2z^2$

5. $\dfrac{\partial^2 z}{\partial x^2}=-y\,e^x$, $\dfrac{\partial^2 z}{\partial x\partial y}=e^y-e^x$, $\dfrac{\partial^2 z}{\partial y^2}=x\,e^y$

7. 27, 10, $x^2e^{xy}\big(xz\sin xz-(3+xy)\cos xz\big)$

19. $u(x,y,z,t)=t^{-3/2}e^{-(x^2+y^2+z^2)/4t}$

Section 12.5 (page 741)

1. $\dfrac{\partial w}{\partial t}=f_1g_2+f_2h_2+f_3k_2$

3. $\dfrac{\partial z}{\partial u}=g_1h_1+g_2f'h_1$

5. $\dfrac{dw}{dz}=f_1g_1h'+f_1g_2+f_2h'+f_3$,
$\dfrac{\partial w}{\partial z}\Big|_x=f_2h'+f_3$,
$\dfrac{\partial w}{\partial z}\Big|_{x,y}=f_3$

7. $\dfrac{\partial z}{\partial x}=\dfrac{-5y}{13x^2-2xy+2y^2}$

9. $2f_1(2x,3y)$

11. $2x\,f_2(y^2,x^2)$

13. $dT/dt=e^{-t}\big(f'(t)-f(t)\big)$; $dT/dt=0$ if $f(t)=e^t$: in this case the decrease in T with time (at fixed depth) is exactly balanced by the increase in T with depth.

15. $4f_{11}+12f_{12}+9f_{22}$, $6f_{11}+5f_{12}-6f_{22}$, $9f_{11}-12f_{12}+4f_{22}$

17. $f_1\cos s-f_2\sin s+f_{11}t\cos s\sin s+f_{12}t(\cos^2 s-\sin^2 s)-f_{22}t\sin s\cos s$

19. $f_2 + 2y^2 f_{12} + xyf_{22} - 4xyf_{31} - 2x^2 f_{32}$;
all derivatives at $(y^2, xy, -x^2)$

27. $\sum_{i,j=1}^{n} x_i x_j f_{ij}(x_1, \cdots, x_n) = k(k-1) f(x_1, \cdots, x_n)$

31. $u(x, y) = f(x + ct)$

Section 12.6 (page 750)

1. 6.9

3. 0.0814

5. 2.967

7. (a) 3%, (b) 2%, (c) 1%

9. 8.88 ft^2

11. 169 m, 24 m, most sensitive to angle at B

13. $\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

15. $\begin{pmatrix} 2x & z & y \\ -\ln z & 2y & -x/z \end{pmatrix}$, $(5.99, 3.98)$

Section 12.7 (page 761)

1. $4\mathbf{i} + 2\mathbf{j}$; $z = 4x + 2y - 3$; $2x + y = 3$

3. $(3\mathbf{i} - 4\mathbf{j})/25$; $3x - 4y - 25z + 10 = 0$;
$3x - 4y + 5 = 0$

5. $(2\mathbf{i} - 4\mathbf{j})/5$; $2x - 4y - 5z = 10 - 5\ln 5$; $x - 2y = 5$

7. $x + y - 3z = -3$

9. $\sqrt{3}y + z = \sqrt{3} + \pi/3$

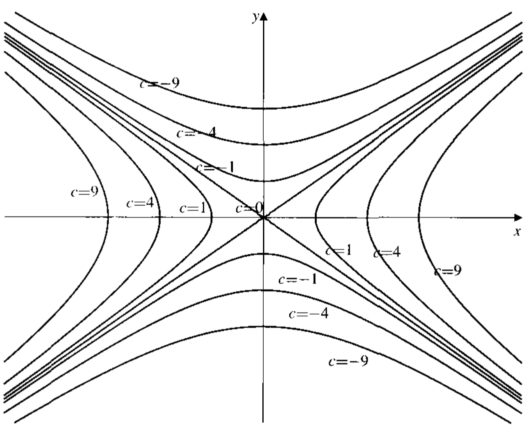
11. $\dfrac{4}{\sqrt{5}}$

13. $1 - 2\sqrt{3}$

17. in directions making angles $-30°$ or $-150°$ with positive x-axis; no; $-\mathbf{j}$.

19. $7\mathbf{i} - \mathbf{j}$

21. a)



b) in direction $-\mathbf{i} - \mathbf{j}$
c) $4\sqrt{2}k$ deg/unit time
d) $12k/\sqrt{5}$ deg/unit time
e) $x^2 y = -4$

23. $3x^2 - 2y^2 = 10$

25. $-4/3$

27. $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

33. $D_{\mathbf{v}}(D_{\mathbf{v}} f) = v_1^2 f_{11} + v_2^2 f_{22} + v_3^2 f_{33} + 2v_1 v_2 f_{12} + 2v_1 v_3 f_{13}$
$+ 2v_2 v_3 f_{23}$.
This is the second derivative of f as measured by an observer moving with velocity $\mathbf{v}$.

35. $\dfrac{\partial^2 T}{\partial t^2} + 2D_{\mathbf{v}(t)}\left(\dfrac{\partial T}{\partial t}\right) + D_{\mathbf{a}(t)}T + D_{\mathbf{v}(t)}(D_{\mathbf{v}(t)}T)$

Section 12.8 (page 772)

1. $-\dfrac{x^4 + 3xy^2}{y^3 + 4x^3 y}$, $y \neq 0$, $y^2 \neq -4x^3$

3. $\dfrac{3xy^4 + xz}{xy - 2y^2 z}$, $y \neq 0$, $x \neq 2yz$

5. $\dfrac{x - 2t^2 w}{2xy^2 - w}$, $w \neq 2xy^2$

7. $-\dfrac{\partial G/\partial x}{\partial G/\partial u}$, $\dfrac{\partial G}{\partial u} \neq 0$

9. $-\dfrac{v^2 H_2 + wH_3}{u^2 H_1 + tH_3}$, $u^2 H_1 + tH_3 \neq 0$,
all derivatives at $(u^2 w, v^2 t, wt)$

11. $\dfrac{2w - 4y}{4x - w}$, $4x \neq w$

13. $\dfrac{1}{6}, \dfrac{1}{2}, \dfrac{1}{6}, -\dfrac{1}{2}, -\dfrac{1}{6}$

15. r; all points except the origin

17. $-3/2$

19. $-\dfrac{\partial(F, G, H)}{\partial(y, z, w)} \Big/ \dfrac{\partial(F, G, H)}{\partial(x, z, w)}$

21. 15; $-\dfrac{\partial(F, G, H)}{\partial(x_2, x_3, x_5)} \Big/ \dfrac{\partial(F, G, H)}{\partial(x_1, x_3, x_5)}$

23. $2(u + v)$, -2, 0

Section 12.9 (page 780)

1. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n y^{2n}}{2^{n+1}}$

3. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}(y+1)^{2n+1}}{2n+1}$

5. $\displaystyle\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} x^{2k} y^{2n-2k}$

7. $\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{2}(y-1) + \frac{1}{8}(x-2)^2$
$-\frac{1}{2}(x-2)(y-1) + \frac{1}{2}(y-1)^2 - \frac{1}{16}(x-2)^3$
$+\frac{3}{8}(x-2)^2(y-1) - \frac{3}{4}(x-2)(y-1)^2 + \frac{1}{2}(y-1)^3$

9. $x + y^2 - \frac{x^3}{3}$

11. $1 - (y-1) + (y-1)^2 - \frac{1}{2}(x - \frac{\pi}{2})^2$

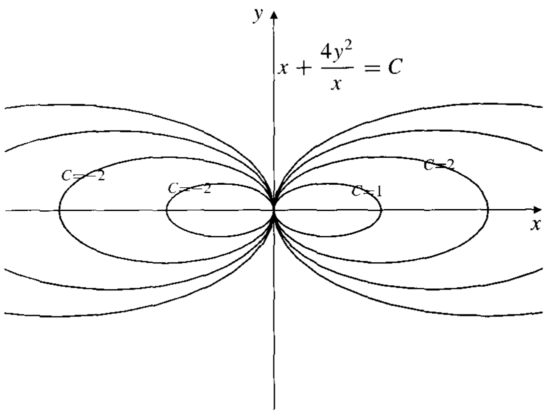
13. $-x - x^2 - (5/6)x^3$

15. $-\dfrac{x}{3} - \dfrac{2y}{3} - \dfrac{2x^2}{27} - \dfrac{8xy}{27} - \dfrac{8y^2}{27}$

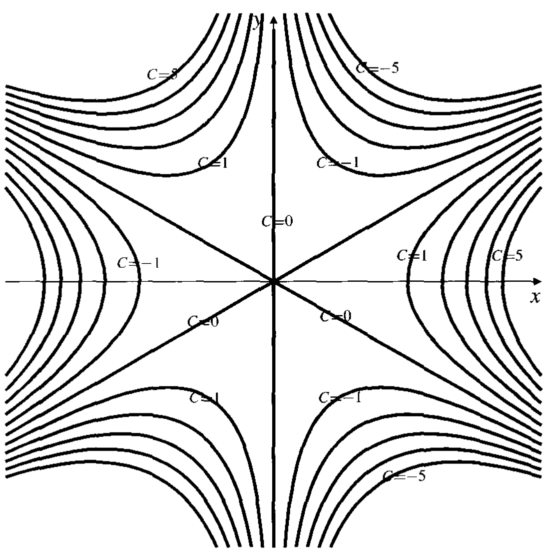
17. $\dfrac{[(2n)!]^3}{(n!)^2}$

Review Exercises (page 780)

1.



3.



5. cont. except on lines $x = \pm y$; can be extended to $x = y$ except at the origin; if $f(0, 0) = 0$ then $f_1(0, 0) = f_2(0, 0) = 1$

7. (a) $ax + by + 4cz = 16$,
(b) the circle $z = 1, x^2 + y^2 = 12$, (c) $\pm(2, 2, \sqrt{2})$

9. $7{,}500$ m^2, 7.213%

11. (a) $-1/\sqrt{2}$, (b) dir. of $\pm(\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, (c) dir. of $-7\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$

15. (a) $\partial u/\partial x = -5$, $\partial u/\partial y = 1$, (b) -1.13

Chapter 13
Applications of Partial Derivatives

Section 13.1 (page 791)

1. $(2, -1)$, loc. (abs) min.
3. $(0, 0)$, saddle pt; $(1, 1)$, loc. min.
5. $(-4, 2)$, loc. max.
7. $(0, n\pi)$, $n = 0, \pm 1, \pm 2, \cdots$, all saddle points
9. $(0, a)$, $(a > 0)$, loc min; $(0, a)$, $(a < 0)$, loc max; $(0, 0)$ saddle point; $(\pm 1, 1/\sqrt{2})$, loc. (abs) max; $(\pm 1, -1/\sqrt{2})$, loc. (abs) min.
11. $(3^{-1/3}, 0)$, saddle pt.
13. $(-1, -1)$, $(1, -1)$, $(-1, 1)$, saddle pts; $(-3, -3)$, loc. min.
15. $(1, 1, \frac{1}{2})$, saddle pt.
17. $(0, 0)$, saddle pt; $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, loc. (abs) max; $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, loc. (abs) min.
19. max $e^{-3/2}/2\sqrt{2}$, min $-e^{-3/2}/2\sqrt{2}$; f is continuous everywhere, and $f(x, y, z) \to 0$ as $x^2 + y^2 + z^2 \to \infty$.
21. $L^3/108$ cu. un.
23. $8abc/(3\sqrt{3})$ cu. units
25. CPs are $(\sqrt{\ln 3}, -\sqrt{\ln 3})$ and $(-\sqrt{\ln 3}, \sqrt{\ln 3})$.
27. f does not have a local minimum at $(0, 0)$; the second derivative test is inconclusive $(B^2 = AC)$.
29. $A > 0$, $\begin{vmatrix} A & E \\ E & B \end{vmatrix} > 0$, $\begin{vmatrix} A & E & F \\ E & B & H \\ F & H & C \end{vmatrix} > 0$,
$\begin{vmatrix} A & E & F & G \\ E & B & H & I \\ F & H & C & J \\ G & I & J & D \end{vmatrix} > 0$

Section 13.2 (page 797)

1. max $5/4$, min -2
3. max $(\sqrt{2} - 1)/2$, min $-(\sqrt{2} + 1)/2$.
5. max $2/3\sqrt{3}$, min 0
7. max 1, min -1
9. max $1/\sqrt{e}$, min $-1/\sqrt{e}$
11. max 4/9, min $-4/9$
13. no limit; yes, max $f = e^{-1}$ (at all points of the curve $xy = 1$)
15. \$625,000, \$733,333
17. max 37/2 at (7/4,5)
19. 6667 kg deluxe, 6667 kg standard

Section 13.3 (page 807)

1. 84,375
3. 1 unit

5. max 4 units, min 2 units

7. $a = \pm\sqrt{3},\ b = \pm 2\sqrt{3},\ c = \pm\sqrt{3}$

9. max 8, min −8

11. max 2, min −2

13. max 7, min −1

15. $\dfrac{2\sqrt{6}}{3}$ units

17. $\pm\sqrt{\dfrac{3n(n+1)}{2(2n+1)}}$

19. $\frac{1}{6} \times \frac{1}{3} \times \frac{2}{3}$

21. width = $\left(\dfrac{2V}{15}\right)^{1/3}$, depth = 3×width, height = $\dfrac{5}{2}$×width

23. max 1, min $-\frac{1}{2}$

27. method will not fail if $\nabla f = \mathbf{0}$ at extreme point; but we will have $\lambda = 0$.

Section 13.4 (page 814)

1. at $(\bar{x}, \bar{y})$ where $\bar{x} = \left(\sum_{i=1}^{n} x_i\right)/n,\ \bar{y} = \left(\sum_{i=1}^{n} y_i\right)/n$

3. $a = \left(\sum_{i=1}^{n} y_i e^{x_i}\right) \Big/ \left(\sum_{i=1}^{n} e^{2x_i}\right)$

5. If $A = \sum x_i^2$, $B = \sum x_i y_i$, $C = \sum x_i$, $D = \sum y_i^2$, $E = \sum y_i$, $F = \sum x_i z_i$, $G = \sum y_i z_i$, and $H = \sum z_i$, then

$$\Delta = \begin{vmatrix} A & B & C \\ B & D & E \\ C & E & n \end{vmatrix}, \quad a = \frac{1}{\Delta}\begin{vmatrix} F & B & C \\ G & D & E \\ H & E & n \end{vmatrix},$$

$$b = \frac{1}{\Delta}\begin{vmatrix} A & F & C \\ B & G & E \\ C & H & n \end{vmatrix}, \quad c = \frac{1}{\Delta}\begin{vmatrix} A & B & F \\ B & D & G \\ C & E & H \end{vmatrix}$$

7. Use linear regression to fit $\eta = a + bx$ to the data $(x_i, \ln y_i)$. Then $p = e^a$, $q = b$. These are not the same values as would be obtained by minimizing the expression $\sum (y_i - pe^{qx_i})^2$.

9. Use linear regression to fit $\eta = a + b\xi$ to the data $\left(x_i, \dfrac{y_i}{x_i}\right)$. Then $p = a, q = b$. Not the same as minimizing $\sum (y_i - px_i - qx_i^2)^2$.

11. Use linear regression to fit $\eta = a + b\xi$ to the data $\left(e^{-2x_i}, \dfrac{y_i}{e^{x_i}}\right)$. Then $p = a, q = b$. Not the same as minimizing $\sum (y_i - pe^{x_i} - qe^{-x_i})^2$. Other answers possible.

13. If $A = \sum x_i^4$, $B = \sum x_i^3$, $C = \sum x_i^2$, $D = \sum x_i$, $H = \sum x_i^2 y_i$, $I = \sum x_i y_i$, and $J = \sum y_i$, then

$$\Delta = \begin{vmatrix} A & B & C \\ B & C & D \\ C & D & n \end{vmatrix}, \quad a = \frac{1}{\Delta}\begin{vmatrix} H & B & C \\ I & C & D \\ J & D & n \end{vmatrix},$$

$$b = \frac{1}{\Delta}\begin{vmatrix} A & H & C \\ B & I & D \\ C & J & n \end{vmatrix}, \quad c = \frac{1}{\Delta}\begin{vmatrix} A & B & H \\ B & C & I \\ C & D & J \end{vmatrix}$$

15. $a = 5/6,\ I = 1/252$

17. $a = 15/16,\ b = -1/16,\ I = 1/448$

19. $a = \frac{20}{\pi^5}(\pi^2 - 16), \qquad b = \frac{12}{\pi^4}(20 - \pi^2)$

21. $a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx\,dx, \quad (k = 0, 1, 2, \cdots)$

23. $\pi - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}; \quad -x$

Section 13.5 (page 824)

1. $\frac{(-1)^n n!}{(x+1)^{n+1}}$

3. $2\sqrt{\pi}(\sqrt{y} - \sqrt{x})$

5. $\frac{2x}{(1+x^2)^2}; \quad \frac{(6x^2-2)}{(1+x^2)^3}$

7. $\frac{\pi}{2x}$, assume $x > 0$; $\frac{\pi}{4x^3}$; $\frac{3\pi}{16x^5}$

9. $n!$

11. $f(x) = \int_0^x e^{-t^2/2}\,dt$

13. $y = x^2$

15. $x^2 + y^2 = 1$

17. $y = x - \frac{1}{4}$

19. no

21. no; a line of singular points

23. $x^2 + y^2 + z^2 = 1$

25. $y = x - \epsilon\,\sin(\pi x) + \frac{\pi\epsilon^2}{2}\,\sin(2\pi x) + \cdots$

27. $y = \frac{1}{2} - \frac{2}{5}\epsilon x - \frac{16}{125}\epsilon^2 x^2 + \cdots$

29. $x \approx 1 - \frac{1}{100e} - \frac{1}{30000e^2}, \quad y \approx 1 - \frac{1}{30000e^2}$

Section 13.6 (page 828)

1. (0.797105, 2.219107)

3. (±0.2500305, ±3.9995115), (±1.9920783, ±0.5019883)

5. (0.3727730, 0.3641994), (−1.4141606, −0.9877577)

7. $x = x_0 - \frac{\Delta_1}{\Delta}, \quad y = y_0 - \frac{\Delta_2}{\Delta}, \quad z = z_0 - \frac{\Delta_3}{\Delta}$, where $\Delta = \left.\dfrac{\partial(f, g, h)}{\partial(x, y, z)}\right|_{(x_0, y_0, z_0)}$ and Δ_i is Δ with the ith column replaced with $\begin{matrix} f \\ g \\ h \end{matrix}$

9. 18 iterations near (0, 0), 4 iterations near (1, 1); the two curves are tangent at (0,0), but not at (1,1).

Section 13.7 (page 833)

1. (±.45304, .81204, ±.36789), (±.96897, .17751, ±.17200)

3. local and absolute max .81042 at (−.33853, −.52062); local and absolute min −.66572 at (.13319, .53682)

5. −4.5937

Review Exercises (page 834)

1. (0, 0) saddle pt., (1, −1) loc. min.

3. (2/3, 4/3) loc. min; (2, −4) and (−1, 2) saddle points

5. yes, 2, on the sphere $x^2 + y^2 + z^2 = 1$

7. max $1/(4e)$, min $-1/(4e)$

9. (a) $L^2/48\ \text{cm}^2$, (b) $L^2/16\ \text{cm}^2$

11. 4π sq. units **13.** 16π cu. units

15. 1,688 widgets, \$2.00 each

17. $y \approx -2x - \epsilon x e^{-2x} + \epsilon^2 x^2 e^{-4x}$

Challenging Problems (page 835)

3. $\frac{1}{2}\ln(1+x^2)\tan^{-1}x$

Chapter 14 Multiple Integration

Section 14.1 (page 841)

1. 15 **3.** 21
5. 15 **7.** 96
9. 80 **11.** 36.6258
13. 20 **15.** 0
17. 5π **19.** $\frac{\pi a^3}{3}$
21. $\frac{1}{6}$

Section 14.2 (page 849)

1. 5/24 **3.** 4
5. $\frac{ab(a^2+b^2)}{3}$ **7.** π
9. $\frac{3}{56}$ **11.** $\frac{33}{8}\ln 2 - \frac{45}{16}$
13. $\frac{e-2}{2}$

15. $\frac{1}{2}\left(1-\frac{1}{e}\right)$; region is a triangle with vertices (0,0), (1,0) and (1,1)

17. $\frac{\pi}{4\lambda}$; region is a triangle with vertices (0,0), (0,1) and (1,1)

19. 1/4 cu. units **21.** 1/3 cu. units
23. $\ln 2$ cu. units **25.** $\frac{\pi}{2\sqrt{2}}$ cu. units
27. $\frac{16a^3}{3}$ cu. units

Section 14.3 (page 855)

1. converges to 1 **3.** converges to $\pi/2$
5. diverges to ∞ **7.** converges to 4
9. converges to $1-\frac{1}{e}$ **11.** diverges to ∞
13. converges to $2\ln 2$ **15.** $k > a - 1$
17. $k < -1 - a$

19. $k > -\frac{1+a}{1+b}$ (provided $b > -1$)

21. $\frac{1}{2}$, $-\frac{1}{2}$ (different answers are possible because the *double integral* does not exist.)

23. $\frac{a^2}{3}$ **25.** $\frac{4\sqrt{2}a}{3\pi}$
27. yes, $1/(2\pi)$

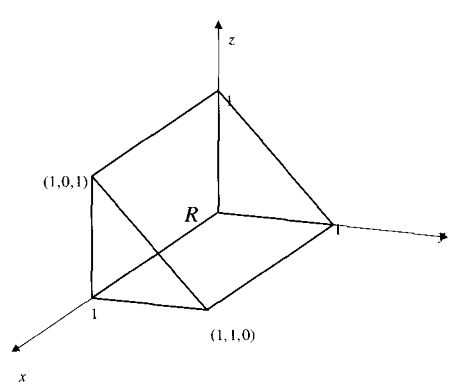
Section 14.4 (page 865)

1. $\pi a^4/2$ **3.** $2\pi a$
5. $\pi a^4/4$ **7.** $a^3/3$
9. $\pi(e^{a^2}-1)/4$ **11.** $\frac{(\sqrt{3}+1)a^3}{6}$
13. $\frac{1}{3}$ **15.** $\frac{2a}{3}$
17. $k < 1$; $\frac{\pi}{1-k}$ **19.** $\frac{a^4}{16}$
21. $\frac{2\pi}{3}$ cu. units **23.** $\frac{4\pi(2\sqrt{2}-1)a^3}{3}$ cu. units
25. $16[1-(1/\sqrt{2})]a^3$ cu. units
27. $1-\frac{4\sqrt{2}}{3\pi}$ units **29.** $\frac{4}{3}\pi abc$ cu. units
31. $2a\sinh a$ **33.** $\frac{3\ln 2}{2}$ sq. units
35. $\frac{1}{4}(e-e^{-1})$

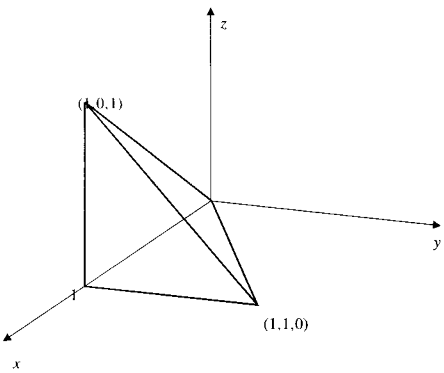
Section 14.5 (page 873)

1. $8abc$ **3.** 16π
5. 2/3 **7.** 1/15
9. $2/(3\pi)$ **11.** $\frac{3}{16}\ln 2$
13. $\pi\sqrt{\frac{\pi}{6}}$ **15.** 1/8

17. $\int_0^1 dx \int_0^1 dy \int_0^{1-y} f(x,y,z)\,dz$



19. $\int_0^1 dx \int_0^x dy \int_0^{x-y} f(x, y, z)\, dz$



27. $(e - 1)/3$

29. $\bar{f} = \dfrac{1}{\text{vol}(R)} \iiint_R f\, dV$; 1

Section 14.6 (page 882)

1. Cartesian: $(-\sqrt{3}, 3, 2)$; cylindrical: $[2\sqrt{3}, 2\pi/3, 2]$

3. Cartesian: $(\sqrt{3}, 1, -2)$; spherical: $[2\sqrt{2}, 3\pi/4, \pi/6]$

5. the half-plane $x = 0$, $y > 0$

7. the xy-plane

9. the circular cylinder of radius 4 with axis along the z-axis

11. the xy-plane

13. sphere of radius 1 with centre $(0, 0, 1)$

15. $\frac{2}{3}\pi a^3 \left(1 - \frac{1}{\sqrt{2}}\right)$ cu. units

17. 24π cu. units

19. $(2\pi - \dfrac{32}{9})a^3$ cu. units.

21. $\dfrac{abc}{3} \tan^{-1} \dfrac{a}{b}$ cu. units **23.** $\frac{\pi ab}{2}$ cu. units

25. $\dfrac{8\pi a^5}{15}$

27. $\dfrac{2\pi a^5}{5} \left(1 - \dfrac{c}{\sqrt{c^2 + 1}}\right)$

29. $\dfrac{7\pi}{12}$ **31.** $\frac{ha^3}{12}$; $\frac{\pi a^2 h^2}{48}$

Section 14.7 (page 891)

1. 3π sq. units **3.** $2\pi a^2$ sq. units

5. $24\pi/\sqrt{3}$ sq. units

7. $(5\sqrt{5} - 1)/12$ sq. units

9. 4 sq. units **11.** 5.123

13. $4\pi A \left[a - \sqrt{B} \tan^{-1}\left(\frac{a}{\sqrt{B}}\right)\right]$ units

15. $2\pi k m\delta(h + \sqrt{a^2 + (b - h)^2} - \sqrt{a^2 + b^2})$

17. $2\pi k m\delta(h + \sqrt{a^2 + (b - h)^2} - \sqrt{a^2 + b^2})$

19. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$ **21.** $(\frac{3a}{8}, \frac{3a}{8}, \frac{3a}{8})$

23. The model still involves angular acceleration to spin the ball—it doesn't just fall. Part of the gravitational energy goes to producing this spin even in the limiting case.

25. $I = \pi\delta a^2 h \left(\frac{h^2}{3} + \frac{a^2}{4}\right)$, $\bar{D} = \left(\frac{h^2}{3} + \frac{a^2}{4}\right)^{1/2}$

27. $I = \frac{\pi\delta a^2 h}{3}\left(\frac{2h^2+3a^2}{20}\right)$, $\bar{D} = \left(\frac{2h^2+3a^2}{20}\right)^{1/2}$

29. $I = \frac{5a^5\delta}{12}$, $\bar{D} = \sqrt{\frac{5}{12}}\, a$

31. $I = \frac{8}{3}\delta abc(a^2 + b^2)$, $\bar{D} = \sqrt{\frac{a^2+b^2}{3}}$

33. $m = \frac{4\pi}{3}\delta(a^2 - b^2)^{3/2}$, $I = \frac{1}{5}m(2a^2 + 3b^2)$

35. $\frac{5a^2 g \sin\alpha}{7a^2+3b^2}$

39. The moment of inertia about the line $\mathbf{r}(t) = At\mathbf{i} + Bt\mathbf{j} + Ct\mathbf{k}$ is

$$\frac{1}{A^2 + B^2 + C^2}\big((B^2 + C^2)P_{xx} + (A^2 + C^2)P_{yy} + (A^2 + B^2)P_{zz} - 2ABP_{xy} - 2ACP_{xz} - 2BCP_{yz}\big).$$

Review Exercises (page 892)

1. 3/10 **3.** ln 2

5. $k = 1/\sqrt{3}$

7. $\displaystyle\int_0^1 dx \int_x^1 dy \int_y^1 f(x, y, z)\, dz$

9. $(1 - e^{-a^2})/(2a)$ **11.** $\dfrac{8\pi}{15}(18\sqrt{6} - 41)a^5$

13. vol = 7/12, $\bar{z}$ = 11/28 **15.** 17/24

17. $\displaystyle\frac{1}{6}\int_0^{\pi/2} \left[(1 + 16\cos^2\theta)^{3/2} - 1\right] d\theta \approx 7.904$ sq. units

Challenging Problems (page 893)

1. $\pi abc\left(\dfrac{2}{3} - \dfrac{8}{9\sqrt{3}}\right)$ cu. units

3. (b) (i) $\sum_{n=1}^{\infty}(-1)^{n-1}1/n^2$, (ii) $\sum_{n=1}^{\infty} 1/n^3$, (iii) $\sum_{n=1}^{\infty}(-1)^{n-1}1/n^3$

5. $4 - \tan^{-1}(\sqrt{2}) + \dfrac{32}{3}\tan^{-1}\left(\dfrac{5}{\sqrt{2}}\right) - \dfrac{4}{3}(7\pi + 2\sqrt{2})$ ≈ 18.9348 cu. units

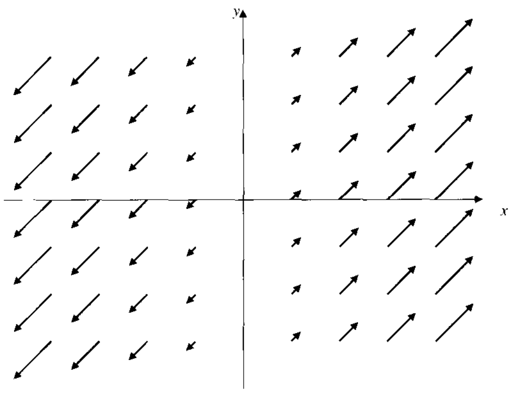
7. $a^3/210$ cu. units

Chapter 15

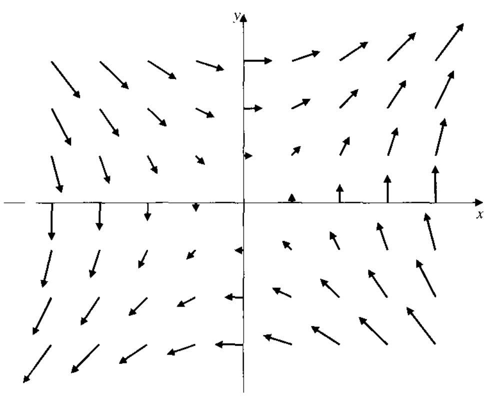
Vector Fields

Section 15.1 (page 900)

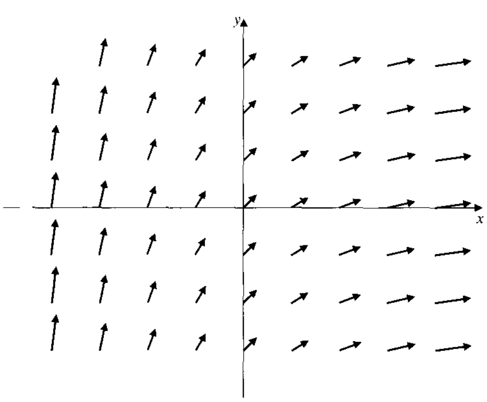
1. field lines: $y = x + C$



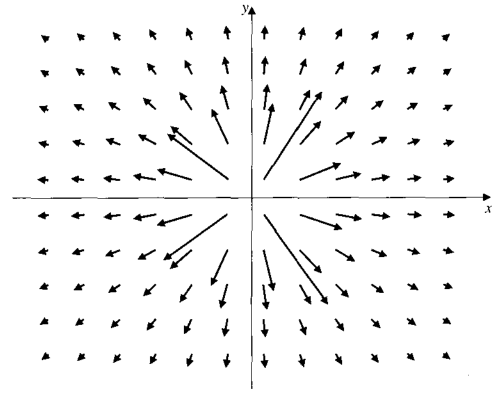
3. field lines: $y^2 = x^2 + C$



5. field lines: $y = -\frac{1}{2}e^{-2x} + C$



7. field lines: $y = Cx$



9. streamlines are lines parallel to $\mathbf{i} - \mathbf{j} - \mathbf{k}$

11. streamlines: $x^2 + y^2 = a^2$, $x = a\sin(z - b)$ (spirals)

13. $y = C_1 x$, $2x = z^2 + C_2$

15. $y = Ce^{1/x}$ **17.** $r = \theta + C$

19. $r = C\theta^2$

Section 15.2 (page 909)

1. conservative; $\dfrac{x^2}{2} - y^2 + \dfrac{3z^2}{2}$

3. not conservative

5. conservative; $x^2y + y^2z - z^2x$

7. $-2\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^4}$

9. $(x^2 + y^2)/z$; equipotential surfaces are paraboloids $z = C(x^2+y^2)$; field lines are ellipses $x^2+y^2+2z^2 = A$, $y = Bx$ in vertical planes through the origin

11. $\mathbf{v} = \dfrac{m(x\mathbf{i} + y\mathbf{j} + (z - \ell)\mathbf{k})}{[x^2 + y^2 + (z - \ell)^2]^{3/2}} + \dfrac{m(x\mathbf{i} + y\mathbf{j} + (z + \ell)\mathbf{k})}{[x^2 + y^2 + (z + \ell)^2]^{3/2}}$, $\mathbf{v} = \mathbf{0}$ only at the origin; $\mathbf{v}(x, y, 0) = \dfrac{2m(x\mathbf{i} + y\mathbf{j})}{(x^2 + y^2 + \ell^2)^{3/2}}$; speed maximum on the circle $x^2 + y^2 = \ell^2/2$, $z = 0$

15. $\phi = -\frac{\mu y}{r^2}$, $\mathbf{F} = \frac{\mu(2xy\mathbf{i}+(y^2-x^2)\mathbf{j})}{r^4}$, $(r^2 = x^2 + y^2)$

21. $\phi = \frac{1}{2}r^2 \sin 2\theta$

Section 15.3 (page 915)

1. $\frac{a^2}{2}\left(\sqrt{2} + \ln(1 + \sqrt{2})\right)$ **3.** 8 g

5. $\frac{\delta}{6}\left((2e^{4\pi} + 1)^{3/2} - 3^{3/2}\right)$

7. $3\sqrt{14}$

9. $m = 2\sqrt{2}\pi^2$, $(0, -1/\pi, 4\pi/3)$

11. $(e^6 + 3e^4 - 3e^2 - 1)/(3e^3)$

13. $\left(\sqrt{2} + \ln(\sqrt{2} + 1)\right)a^2/2$

15. $\pi/\sqrt{2}$

17. $4\sqrt{b^2+c^2}\,E\left(\sqrt{\dfrac{b^2-a^2}{b^2+c^2}}\right)$;

$\sqrt{b^2+c^2}\,E\left(\sqrt{\dfrac{b^2-a^2}{b^2+c^2}},\,T\right)$

Section 15.4 (page 923)

1. $-1/4$ **3.** $1/2$

5. 0 **7.** $19/2$

9. $e^{1+(\pi/4)}$

11. $A=2,\ B=3;\ 4\ln 2-\frac{1}{2}$

13. $-13/2$ **15.** (a) πa^2, (b) $-\pi a^2$

17. (a) $\dfrac{\pi a^2}{2}$, (b) $-\dfrac{\pi a^2}{2}$ **19.** (a) $ab/2$, (b) $-ab/2$

23. The plane with origin removed is not simply connected.

Section 15.5 (page 935)

1. $dS = ds\,dz = \sqrt{(g(\theta))^2+(g'(\theta))^2}\,d\theta\,dz$

3. $\frac{\pi ab\sqrt{A^2+B^2+C^2}}{|C|}$ sq. units $(C \neq 0)$

5. (a) $dS = |\nabla F/F_2|\,dx\,dz$, (b) $dS = |\nabla F/F_1|\,dy\,dz$

7. $\frac{\pi}{8}$ **9.** $16a^2$ sq. units

13. 2π **15.** $1/96$

17. $\pi(3e+e^3-4)/3$

19. $2\pi a^2 + \dfrac{2\pi ac^2}{\sqrt{a^2-c^2}}\ln\left(\dfrac{a+\sqrt{a^2-c^2}}{c}\right)$ sq. units

21. $2\pi\sqrt{A^2+B^2+C^2}/|D|$

23. one-third of the way from the base to the vertex on the axis

25. $2\pi k\sigma m\left(\frac{1}{\sqrt{a^2+(b-h)^2}} - \frac{1}{\sqrt{a^2+b^2}}\right)$

27. $I = \frac{8}{3}\pi\sigma a^4;\ \bar{D} = \sqrt{\frac{2}{3}}\,a$

29. $\frac{3}{5}\,g\sin\alpha$

Section 15.6 (page 943)

1. 6 **3.** $3abc$

5. $\pi(3a^2-4ab+b^2)/2$ **7.** 4π

9. $2\sqrt{2}\pi$ **11.** $4\pi/3$

13. $4\pi m$ **15.** a) $2\pi a^2$, b) 8

Review Exercises (page 944)

1. $(3e/2)-(3/(2e))$ **3.** $8\sqrt{2}/15$

5. 1

7. (a) $6\pi mgb$, (b) $6\pi R\sqrt{a^2+b^2}$

9. (b) e^2 **11.** $(x\mathbf{i}-y\mathbf{j})/\sqrt{x^2+y^2}$

Challenging Problems (page 944)

1. centroid $(0,0,2/\pi)$; upper half of the surface of the torus obtained by rotating the circle $(x-2)^2+z^2=1$, $y=0$, about the z-axis

Chapter 16
Vector Calculus

Section 16.1 (page 953)

1. $\mathbf{div\,F}=2,\ \mathbf{curl\,F}=\mathbf{0}$

3. $\mathbf{div\,F}=0,\ \mathbf{curl\,F}=-\mathbf{i}-\mathbf{j}-\mathbf{k}$

5. $\mathbf{div\,F}=1,\ \mathbf{curl\,F}=-\mathbf{j}$

7. $\mathbf{div\,F}=f'(x)+g'(y)+h'(z)$, $\mathbf{curl\,F}=\mathbf{0}$

9. $\mathbf{div\,F}=\cos\theta\left(1+\dfrac{1}{r}\cos\theta\right)$;

$\mathbf{curl\,F}=-\sin\theta\left(1+\dfrac{1}{r}\cos\theta\right)\mathbf{k}$

11. $\mathbf{div\,F}=0$; $\mathbf{curl\,F}=(1/r)\mathbf{k}$

Section 16.2 (page 961)

7. **div F** can have any value, **curl F** must be normal to **F**

9. $f(r)=Cr^{-3}$

15. If $\mathbf{F}=\nabla\phi$ and $\mathbf{G}=\nabla\psi$ then $\nabla\times(\phi\nabla\psi)=\mathbf{F}\times\mathbf{G}$.

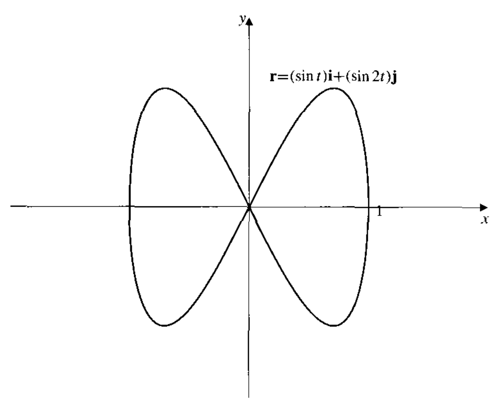
17. $\mathbf{G}=ye^{2z}\mathbf{i}+xye^{2z}\mathbf{k}$ is one possible vector potential.

Section 16.3 (page 965)

1. πa^2-4a^3 **3.** 9

5. $\dfrac{3\pi ab}{8}$ sq. units

7. 0



Section 16.4 (page 971)

1. $4\pi a^3$ **3.** $(4/3)\pi a^3$

5. 360π **7.** $81/4$

11. $\frac{2}{3}\pi a^2 b + \frac{3}{10}\pi a^4 b + \pi a^2$

13. (a) $12\sqrt{3}\pi a^3$, (b) $-4\sqrt{3}\pi a^3$, (c) $16\sqrt{3}\pi a^3$

15. $(6 + 2\bar{x} + 4\bar{y} - 2\bar{z})V$ **17.** $9\pi a^2$

Section 16.5 (page 977)

1. $1/2$ **3.** $-3\pi a^2$

7. 9π

9. $\alpha = -\frac{1}{2}$, $\beta = -3$, $I = -\frac{3}{8}\pi$

11. yes, $\phi\nabla\psi$

Section 16.7 (page 996)

1. $\nabla f = \theta z\hat{\mathbf{r}} + z\hat{\boldsymbol{\theta}} + r\theta\mathbf{k}$

3. $\mathbf{div\,F} = 2$, $\mathbf{curl\,F} = \mathbf{0}$

5. $\mathbf{div\,F} = \dfrac{2\sin\phi}{\rho}$, $\mathbf{curl\,F} = -\dfrac{\cos\phi}{\rho}\hat{\boldsymbol{\theta}}$

7. $\mathbf{div\,F} = 0$, $\mathbf{curl\,F} = \cot\phi\hat{\boldsymbol{\rho}} - 2\hat{\boldsymbol{\phi}}$

9. scale factors: $h_u = \left|\dfrac{\partial\mathbf{r}}{\partial u}\right|$, $h_v = \left|\dfrac{\partial\mathbf{r}}{\partial v}\right|$

local basis: $\hat{\mathbf{u}} = \dfrac{1}{h_u}\dfrac{\partial\mathbf{r}}{\partial u}$, $\hat{\mathbf{v}} = \dfrac{1}{h_v}\dfrac{\partial\mathbf{r}}{\partial v}$

area element: $dA = h_u h_v\,du\,dv$

11. $\nabla f(r,\theta) = \dfrac{\partial f}{\partial r}\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial r}{\partial\theta}\hat{\boldsymbol{\theta}}$

$\nabla\bullet\mathbf{F}(r,\theta) = \dfrac{\partial F_r}{\partial r} + \dfrac{1}{r}F_r + \dfrac{1}{r}\dfrac{\partial F_\theta}{\partial\theta}$

$\nabla\times\mathbf{F}(r,\theta) = \left(\dfrac{\partial F_\theta}{\partial r} + \dfrac{1}{r}F_\theta - \dfrac{1}{r}\dfrac{\partial F_r}{\partial\theta}\right)\mathbf{k}$

13. u-surfaces: vertical elliptic cylinders with focal axes at $x = \pm a$, $y = 0$

v-surfaces: vertical hyperbolic cylinders with focal axes at $x = \pm a$, $y = 0$

z-surfaces: horizontal planes

u-curves: horizontal hyperbolas with foci $x = \pm a$, $y = 0$

v-curves: horizontal ellipses with foci $x = \pm a$, $y = 0$

z-curves: vertical straight lines

15. $$\nabla f = \frac{\partial^2 f}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial f}{\partial\phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

Review Exercises (page 997)

1. 128π **3.** -6

5. $3/4$ **7.** $\lambda = -3$, no

11. the ellipsoid $x^2 + 4y^2 + z^2 = 4$ with outward normal

Challenging Problems (page 998)

1. $\mathbf{div\,v} = 3C$

Appendix I Complex Numbers
(page A-11)

1. $\Re(z) = -5$, $\Im(z) = 2$ **3.** $\Re(z) = 0$, $\Im(z) = -\pi$

5. $|z| = \sqrt{2}$, $\theta = 3\pi/4$ **7.** $|z| = 3$, $\theta = \pi/2$

9. $|z| = \sqrt{5}$, $\theta = \tan^{-1}2$

11. $|z| = 5$, $\theta = \pi + \tan^{-1}(4/3)$

13. $|z| = 2$, $\theta = 11\pi/6$ **15.** $|z| = 3$, $\theta = 4\pi/5$

17. $23\pi/12$ **19.** $4 + 3i$

21. $\frac{\pi\sqrt{3}}{2} + \frac{\pi}{2}i$ **23.** $\frac{1}{4} - \frac{\sqrt{3}}{4}i$

25. $-3 + 5i$ **27.** $2 + i$

29. closed disk, radius 2, centre 0

31. closed disk, radius 5, centre $3 - 4i$

33. closed plane sector lying under $y = 0$ and to the left of $y = -x$

35. 4 **37.** $5 - i$

39. $2 + 11i$ **41.** $-\frac{1}{5} + \frac{7}{5}i$

43. 1

47. $zw = -3 - 3i$, $\dfrac{z}{w} = \dfrac{1+i}{3}$

49. (a) circle $|z| = \sqrt{2}$, (b) no solutions

51. -1, $\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}$

53. $2^{1/6}(\cos\theta + i\sin\theta)$ where $\theta = \pi/4, 11\pi/12, 19\pi/12$

55. $\pm 2^{1/4}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$, $\pm 2^{1/4}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

Appendix IV Differential Equations
(page A-38)

1. 1, linear, homogeneous **3.** 1, nonlinear

5. 2, linear, homogeneous

7. 3, linear, nonhomogeneous

9. 4, linear, homogeneous

11. (a) and (b) are solutions, (c) is not

13. $y_2 = \sin(kx)$, $y = -3(\cos(kx) + (3/k)\sin(kx))$

15. $y = \sqrt{2}(\cos x + 2\sin x)$

17. $y = x + \sin x + (\pi - 1)\cos x$

19. $2\tan^{-1}(y/x) = \ln(x^2 + y^2) + C$

21. $\tan^{-1}(y/x) = \ln|x| + C$

23. $y = x\tan^{-1}(\ln|Cx|)$ **25.** $y^3 + 3y - 3x^2 = 24$

25. $x + y = 4x^2$

27. $4\tan^{-1}\frac{y-1}{x-2} = \ln\left((y-1)^2 + (x-2)^2\right) + C$

29. $e^x \sin y + x^2 + y^2 = C$

31. $x^2 + x + \dfrac{y^2}{x} = C$

33. $e^x + x\ln y + y\ln x - \cos y = C$

35. $\mu(y) = \dfrac{1}{y}, \quad x^2y + 2xy^3 = C$

37. $x\mu'(xy)M + \mu(xy)\dfrac{\partial M}{\partial y} = y\mu'(xy)N + \mu(xy)\dfrac{\partial N}{\partial x}$

39. (a) 1.97664, (b) 2.187485, (c) 2.306595

41. (a) 2.436502, (b) 2.436559, (c) 2.436563

43. (a) 1.097897, (b) 1.098401

45. $y = 2/(3-2x)$

49. (b) $u = 1/(1-x)$, $v = \tan(x + \frac{\pi}{4})$. $y(x)$ is defined at least on $[0, \pi/4)$ and satisfies $1/(1-x) \le y(x) \le \tan(x+\frac{\pi}{4})$ there.

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