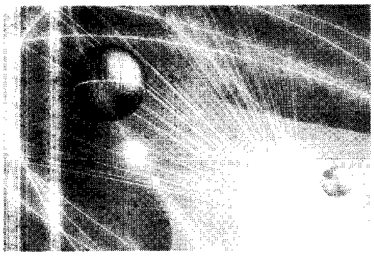

CHAPTER 15

Vector Fields

Introduction This chapter and the next are concerned mainly with vector-valued functions of a vector variable, typically functions whose domains and ranges lie in the plane or in 3-space. Such functions are frequently called *vector fields.* Applications of vector fields often involve integrals taken, not along axes or over regions in the plane or 3-space, but rather over curves and surfaces. We will introduce such line and surface integrals in this chapter. The next chapter will be devoted to developing analogues of the Fundamental Theorem of Calculus for integrals of vector fields.

15.1 Vector and Scalar Fields

A function whose domain and range are subsets of Euclidean 3-space, $\mathbb{R}^3$, is called a **vector field**. Thus, a vector field $\mathbf{F}$ associates a vector $\mathbf{F}(x, y, z)$ with each point (x, y, z) in its domain. The three components of $\mathbf{F}$ are scalar-valued (real-valued) functions $F_1(x, y, z)$, $F_2(x, y, z)$, and $F_3(x, y, z)$, and $\mathbf{F}(x, y, z)$ can be expressed in terms of the standard basis in $\mathbb{R}^3$ as

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}.$$

(Note that the subscripts here represent *components* of a vector, *not* partial derivatives.) If $F_3(x, y, z) = 0$ and F_1 and F_2 are independent of z, then $\mathbf{F}$ reduces to

$$\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$$

and so is called a **plane vector field**, or a vector field in the xy-plane. We will frequently make use of position vectors in the arguments of vector fields. The position vector of (x, y, z) is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and we can write $\mathbf{F}(\mathbf{r})$ as a shorthand for $\mathbf{F}(x, y, z)$. In the context of discussion of vector fields, a scalar-valued function of a vector variable (i.e., a function of several real variables as considered in the context of Chapters 12–14) is frequently called a **scalar field**. Thus, the components of a vector field are scalar fields.

Many of the results we prove about vector fields require that the field be smooth in some sense. We will call a vector field **smooth** wherever its component scalar fields have continuous partial derivatives of all orders. (For most purposes, however, second order would be sufficient.)

Vector fields arise in many situations in applied mathematics. Let us list some:

(a) The gravitational field $\mathbf{F}(x, y, z)$ due to some object is the force of attraction that the object exerts on a unit mass located at position (x, y, z).

(b) The electrostatic force field $\mathbf{E}(x, y, z)$ due to an electrically charged object is the electrical force that the object exerts on a unit charge at position (x, y, z). (The force may be either an attraction or a repulsion.)

(c) The velocity field $\mathbf{v}(x, y, z)$ in a moving fluid (or solid) is the velocity of motion of the particle at position (x, y, z). If the motion is not "steady state," then the conducting medium. *Pressure gradients* provide similar information about the variation of pressure in a fluid such as an air mass or an ocean.

(e) The unit radial and unit transverse vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are examples of vector fields in the xy-plane. Both are defined at all points of the plane except the origin.

Example 1 **(The gravitational field of a point mass)** The gravitational force field due to a point mass m located at point P_0 having position $\mathbf{r}_0$ is

$$\begin{aligned}\mathbf{F}(x, y, z) = \mathbf{F}(\mathbf{r}) &= \frac{-km}{|\mathbf{r} - \mathbf{r}_0|^3}(\mathbf{r} - \mathbf{r}_0)\\ &= -km\frac{(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}}{\left((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right)^{3/2}},\end{aligned}$$

where $k > 0$ is a constant. $\mathbf{F}$ points toward the point $\mathbf{r}_0$ and has magnitude

$$|\mathbf{F}| = km/|\mathbf{r} - \mathbf{r}_0|^2.$$

Some vectors in a plane section of the field are shown graphically in Figure 15.1. Each represents the value of the field at the position of its tail. The lengths of the vectors indicate that the strength of the force increases the closer you get to P_0. ■

Remark The electrostatic field $\mathbf{F}$ due to a point charge q at P_0 is given by the same formula as the gravitational field above, except with $-m$ replaced by q. The reason for the opposite sign is that like charges repel each other whereas masses attract each other.

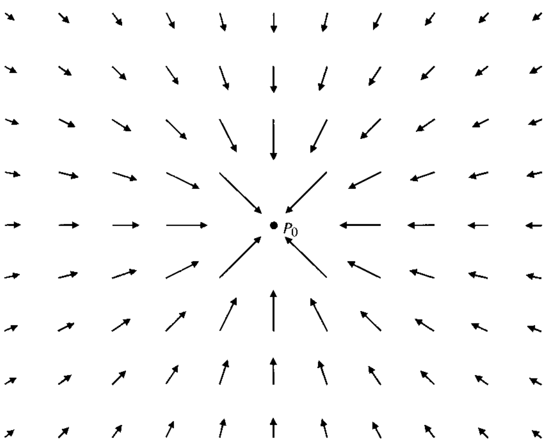

Figure 15.1 The gravitational field of a point mass located at P_0

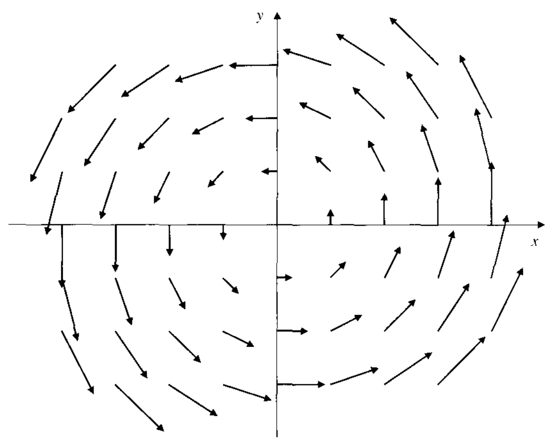

Figure 15.2 The velocity field of a rigid body rotating about the z-axis

Example 2 The velocity field of a solid rotating about the z-axis with angular velocity $\boldsymbol{\Omega} = \Omega\mathbf{k}$ is

$$\mathbf{v}(x, y, z) = \mathbf{v}(\mathbf{r}) = \boldsymbol{\Omega}\times\mathbf{r} = -\Omega y\mathbf{i} + \Omega x\mathbf{j}.$$

Being the same in all planes normal to the z-axis, $\mathbf{v}$ can be regarded as a plane vector field. Some vectors of the field are shown in Figure 15.2.

Field Lines (Integral Curves)

The graphical representations of vector fields such as those shown in Figures 15.1 and 15.2 and the wind velocity field over a hill shown in Figure 15.3 suggest a pattern of motion through space or in the plane. Whether or not the field is a velocity field, we can interpret it as such and ask what path will be followed by a particle, initially at some point, whose velocity is given by the field. The path will be a curve to which the field is tangent at every point. Such curves are called **field lines** or **integral curves** for the given vector field. In the specific case where the vector field gives the velocity in a fluid flow, the field lines are called **streamlines** or **flow lines** of the flow; some of these are shown for the air flow in Figure 15.3. For a force field, the field lines are called **lines of force**.

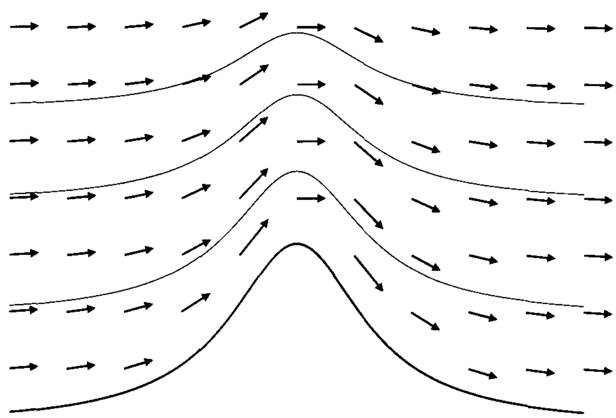

Figure 15.3 The velocity field and some streamlines of wind blowing over a hill

The field lines of $\mathbf{F}$ do not depend on the magnitude of $\mathbf{F}$ at any point but only on the direction of the field. If the field line through some point has parametric equation $\mathbf{r} = \mathbf{r}(t)$, then its tangent vector $d\mathbf{r}/dt$ must be parallel to $\mathbf{F}(\mathbf{r}(t))$ for all t. Thus

$$\frac{d\mathbf{r}}{dt} = \lambda(t)\mathbf{F}\big(\mathbf{r}(t)\big).$$

For *some* vector fields this differential equation can be integrated to find the field lines. If we break the equation into components,

$$\frac{dx}{dt} = \lambda(t)F_1(x, y, z), \quad \frac{dy}{dt} = \lambda(t)F_2(x, y, z), \quad \frac{dz}{dt} = \lambda(t)F_3(x, y, z),$$

we can obtain equivalent differential expressions for $\lambda(t)\,dt$ and hence write the differential equation for the field lines in the form

$$\frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)}.$$

If multiplication of these differential equations by some function puts them in the form

$$P(x)\,dx = Q(y)\,dy = R(z)\,dz,$$

then we can integrate all three expressions to find the field lines.

Example 3 Find the field lines of the gravitational force field of Example 1:

$$\mathbf{F}(x,y,z) = -km\frac{(x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}}{\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}}.$$

Solution The vector in the numerator of the fraction gives the direction of $\mathbf{F}$. Therefore, the field lines satisfy the system

$$\frac{dx}{x-x_0} = \frac{dy}{y-y_0} = \frac{dz}{z-z_0}.$$

Integrating all three expressions leads to

$$\ln|x-x_0| + \ln C_1 = \ln|y-y_0| + \ln C_2 = \ln|z-z_0| + \ln C_3,$$

or, on taking exponentials,

$$C_1(x-x_0) = C_2(y-y_0) = C_3(z-z_0).$$

This represents two families of planes all passing through $P_0 = (x_0, y_0, z_0)$. The field lines are the intersections of planes from each of the families, so they are straight lines through the point P_0. (This is a *two-parameter* family of lines; any one of the constants C_i that is nonzero can be divided out of the equations above.) The nature of the field lines should also be apparent from the plot of the vector field in Figure 15.1.

■

Example 4 Find the field lines of the velocity field $\mathbf{v} = \Omega(-y\mathbf{i} + x\mathbf{j})$ of Example 2.

Solution The field lines satisfy the differential equation

$$\frac{dx}{-y} = \frac{dy}{x}.$$

We can separate variables in this equation to get $x\,dx = -y\,dy$. Integration then gives $x^2/2 = -y^2/2 + C/2$, or $x^2 + y^2 = C$. Thus, the field lines are circles centred at the origin in the xy-plane, as is also apparent from the vector field plot in Figure 15.2. If we regard $\mathbf{v}$ as a vector field in 3-space, we find that the field lines are horizontal circles centred on the z-axis:

$$x^2 + y^2 = C_1, \qquad z = C_2.$$

■

Our ability to find field lines depends on our ability to solve differential equations and, in 3-space, systems of differential equations.

Example 5 Find the field lines of $\mathbf{F} = xz\mathbf{i} + 2x^2z\mathbf{j} + x^2\mathbf{k}$.

Solution The field lines satisfy $\dfrac{dx}{xz} = \dfrac{dy}{2x^2z} = \dfrac{dz}{x^2}$, or, equivalently

$$dy = 2x\,dx \qquad \text{and} \qquad dy = 2z\,dz.$$

The field lines are the curves of intersection of the two families $y = x^2 + C_1$ and $y = z^2 + C_2$ of parabolic cylinders.

Vector Fields in Polar Coordinates

A vector field in the plane can be expressed in terms of polar coordinates in the form

$$\mathbf{F} = \mathbf{F}(r, \theta) = F_r(r, \theta)\hat{\mathbf{r}} + F_\theta(r, \theta)\hat{\boldsymbol{\theta}},$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, defined everywhere except at the origin by

$$\begin{aligned} \hat{\mathbf{r}} &= \quad \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \end{aligned}$$

are unit vectors in the direction of increasing r and θ at $[r, \theta]$. Note that $d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}}$, and that $\hat{\boldsymbol{\theta}}$ is just $\hat{\mathbf{r}}$ rotated 90° counterclockwise. Also note that we are using F_r and F_θ to denote the components of $\mathbf{F}$ with respect to the basis $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}\}$; the subscripts do not indicate partial derivatives. Here, $F_r(r, \theta)$ is called the *radial* component of $\mathbf{F}$, and $F_\theta(r, \theta)$ is called the *transverse* component.

A curve with polar equation $r = r(\theta)$ can be expressed in vector parametric form

$$\mathbf{r} = r\hat{\mathbf{r}},$$

as we did in Section 11.6. This curve is a field line of $\mathbf{F}$ if its differential tangent vector

$$d\mathbf{r} = dr\,\hat{\mathbf{r}} + r\,\frac{d\hat{\mathbf{r}}}{d\theta}\,d\theta = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}$$

is parallel to the field vector $\mathbf{F}(r, \theta)$ at any point except the origin, that is, if $r = f(\theta)$ satisfies the differential equation

$$\frac{dr}{F_r(r, \theta)} = \frac{r\,d\theta}{F_\theta(r, \theta)}.$$

In specific cases we can find the field lines by solving this equation.

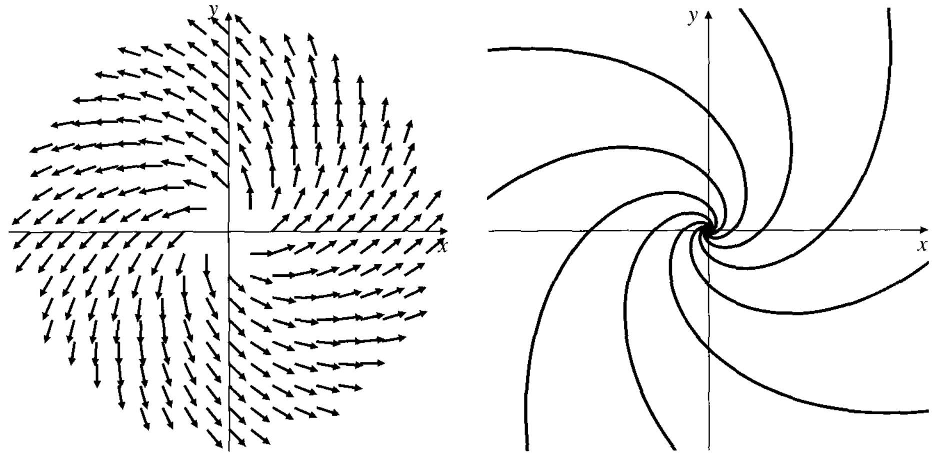

Figure 15.4
(a) The vector field $\mathbf{F} = \hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}$
(b) Field lines of $\mathbf{F} = \hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}$

Example 6 Sketch the vector field $\mathbf{F}(r, \theta) = \hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}$, and find its field lines.

Solution At each point $[r, \theta]$, the field vector bisects the angle between $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, making a counterclockwise angle of 45° with $\hat{\mathbf{r}}$. All of the vectors in the field have the same length, $\sqrt{2}$. Some of the field vectors are shown in Figure 15.4(a). They suggest that the field lines will spiral outward from the origin. Since $F_r(r, \theta) = F_\theta(r, \theta) = 1$ for this field, the field lines satisfy $dr = r\,d\theta$, or, dividing by $d\theta$, $dr/d\theta = r$. This is the differential equation of exponential growth and has solution $r = Ke^\theta$, or, equivalently, $r = e^{\theta+\alpha}$, where $\alpha = \ln K$ is a constant. Several such curves are shown in Figure 15.4(b). ■

Exercises 15.1

In Exercises 1–8, sketch the given plane vector field and determine its field lines.

1. $\mathbf{F}(x, y) = x\mathbf{i} + x\mathbf{j}$
2. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$
3. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$
4. $\mathbf{F}(x, y) = \mathbf{i} + \sin x\,\mathbf{j}$
5. $\mathbf{F}(x, y) = e^x\mathbf{i} + e^{-x}\mathbf{j}$
6. $\mathbf{F}(x, y) = \nabla(x^2 - y)$
7. $\mathbf{F}(x, y) = \nabla \ln(x^2 + y^2)$
8. $\mathbf{F}(x, y) = \cos y\,\mathbf{i} - \cos x\,\mathbf{j}$

In Exercises 9–16, describe the streamlines of the given velocity fields.

9. $\mathbf{v}(x, y, z) = y\mathbf{i} - y\mathbf{j} - y\mathbf{k}$
10. $\mathbf{v}(x, y, z) = x\mathbf{i} + y\mathbf{j} - x\mathbf{k}$
11. $\mathbf{v}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$
12. $\mathbf{v}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j}}{(1 + z^2)(x^2 + y^2)}$
13. $\mathbf{v}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + x\mathbf{k}$
14. $\mathbf{v}(x, y, z) = e^{xyz}(x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k})$
15. $\mathbf{v}(x, y) = x^2\mathbf{i} - y\mathbf{j}$
* 16. $\mathbf{v}(x, y) = x\mathbf{i} + (x + y)\mathbf{j}$ *Hint:* let $y = xv(x)$.

In Exercises 17–20, determine the field lines of the given polar vector fields.

17. $\mathbf{F} = \hat{\mathbf{r}} + r\hat{\boldsymbol{\theta}}$
18. $\mathbf{F} = \hat{\mathbf{r}} + \theta\hat{\boldsymbol{\theta}}$
19. $\mathbf{F} = 2\hat{\mathbf{r}} + \theta\hat{\boldsymbol{\theta}}$
20. $\mathbf{F} = r\hat{\mathbf{r}} - \hat{\boldsymbol{\theta}}$

15.2 Conservative Fields

Since the gradient of a scalar field is a vector field, it is natural to ask whether every vector field is the gradient of a scalar field. Given a vector field $\mathbf{F}(x, y, z)$, does there exist a scalar field $\phi(x, y, z)$ such that

$$\mathbf{F}(x, y, z) = \nabla\phi(x, y, z) = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\,?$$

The answer in general is "no." Only special vector fields can be written in this way.

DEFINITION 1

If $\mathbf{F}(x, y, z) = \nabla\phi(x, y, z)$ in a domain D, then we say that $\mathbf{F}$ is a **conservative** vector field in D, and we call the function ϕ a **(scalar) potential** for $\mathbf{F}$ on D. Similar definitions hold in the plane or in n-space.

Like antiderivatives, potentials are not determined uniquely; arbitrary constants can be added to them. Note that $\mathbf{F}$ is **conservative in a domain** D if and only if $\mathbf{F} = \nabla\phi$ at *every* point of D; the potential ϕ cannot have any singular points in D.

The equation $F_1(x, y, z)\,dx + F_2(x, y, z)\,dy + F_3(x, y, z)\,dz = 0$ is called an **exact** differential equation if the left side is the differential of a scalar function $\phi(x, y, z)$:

$$d\phi = F_1(x, y, z)\,dx + F_2(x, y, z)\,dy + F_3(x, y, z)\,dz.$$

In this case the differential equation has solutions given by $\phi(x, y, z) = C$ (constant). Observe that the differential equation is exact if and only if the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is conservative, and that ϕ is the potential of $\mathbf{F}$.

Being scalar fields rather than vector fields, potentials for conservative vector fields are easier to manipulate algebraically than are the vector fields themselves. For instance, a sum of potential functions is the potential function for the sum of the corresponding vector fields. A vector field can always be computed from its potential function by taking the gradient.

Example 1 **(The gravitational field of a point mass is conservative)** Show that the gravitational field $\mathbf{F}(\mathbf{r}) = -km(\mathbf{r}-\mathbf{r}_0)/|\mathbf{r}-\mathbf{r}_0|^3$ of Example 1 in Section 15.1 is conservative wherever it is defined (i.e., everywhere in $\mathbb{R}^3$ except at $\mathbf{r}_0$), by showing that

$$\phi(x, y, z) = \frac{km}{|\mathbf{r} - \mathbf{r}_0|} = \frac{km}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$

is a potential function for $\mathbf{F}$.

Solution Observe that

$$\frac{\partial\phi}{\partial x} = \frac{-km(x - x_0)}{\Big((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\Big)^{3/2}} = \frac{-km(x - x_0)}{|\mathbf{r} - \mathbf{r}_0|^3} = F_1(\mathbf{r}),$$

and similar formulas hold for the other partial derivatives of ϕ. It follows that $\nabla\phi(x, y, z) = \mathbf{F}(x, y, z)$ for $(x, y, z) \neq (x_0, y_0, z_0)$, and $\mathbf{F}$ is conservative except at $\mathbf{r}_0$.

■

Remark It is not necessary to write the expression $km/|\mathbf{r}-\mathbf{r}_0|$ in terms of the components of $\mathbf{r}-\mathbf{r}_0$ as we did in Example 1 in order to calculate its partial derivatives. Here is a useful formula for the derivative of the length of a vector function $\mathbf{F}$ with respect to a variable x:

$$\frac{\partial}{\partial x}|\mathbf{F}| = \frac{\mathbf{F} \bullet \left(\dfrac{\partial}{\partial x}\mathbf{F}\right)}{|\mathbf{F}|}.$$

To see why this is true, express $|\mathbf{F}| = \sqrt{\mathbf{F} \bullet \mathbf{F}}$ and calculate its derivative using the Chain Rule and the Product Rule:

$$\frac{\partial}{\partial x}|\mathbf{F}| = \frac{\partial}{\partial x}\sqrt{\mathbf{F} \bullet \mathbf{F}} = \frac{1}{2\sqrt{\mathbf{F} \bullet \mathbf{F}}}\,2\mathbf{F} \bullet \left(\frac{\partial}{\partial x}\mathbf{F}\right) = \frac{\mathbf{F} \bullet \left(\frac{\partial}{\partial x}\mathbf{F}\right)}{|\mathbf{F}|}.$$

Compare this with the derivative of an absolute value of a function of one variable:

$$\frac{d}{dx}|f(x)| = \operatorname{sgn}(f(x))\, f'(x) = \frac{f(x)}{|f(x)|}\, f'(x).$$

In the context of Example 1 we have

$$\frac{\partial}{\partial x}\frac{km}{|\mathbf{r}-\mathbf{r}_0|} = \frac{-km}{|\mathbf{r}-\mathbf{r}_0|^2}\frac{\partial}{\partial x}|\mathbf{r}-\mathbf{r}_0| = \frac{-km}{|\mathbf{r}-\mathbf{r}_0|^2}\frac{(\mathbf{r}-\mathbf{r}_0) \bullet \mathbf{i}}{|\mathbf{r}-\mathbf{r}_0|} = \frac{-km(x-x_0)}{|\mathbf{r}-\mathbf{r}_0|^3},$$

with similar expressions for the other partials of $km/|\mathbf{r}-\mathbf{r}_0|$.

Example 2 Show that the velocity field $\mathbf{v} = -\Omega y\mathbf{i} + \Omega x\mathbf{j}$ of rigid body rotation about the z-axis (see Example 2 of Section 15.1) is not conservative if $\Omega \neq 0$.

Solution There are two ways to show that no potential for $\mathbf{v}$ can exist. One way is to try to find a potential $\phi(x, y)$ for the vector field. We require

$$\frac{\partial \phi}{\partial x} = -\Omega y \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = \Omega x.$$

The first of these equations implies that $\phi(x, y) = -\Omega xy + C_1(y)$. (We have integrated with respect to x; the constant can still depend on y.) Similarly, the second equation implies that $\phi(x, y) = \Omega xy + C_2(x)$. Therefore, we must have $-\Omega xy + C_1(y) = \Omega xy + C_2(x)$, or $2\Omega xy = C_1(y) - C_2(x)$ for all (x, y). This is not possible for any choice of the single-variable functions $C_1(y)$ and $C_2(x)$ unless $\Omega = 0$.

Alternatively, if $\mathbf{v}$ has a potential ϕ, then we can form the mixed partial derivatives of ϕ from the two equations above and get

$$\frac{\partial^2 \phi}{\partial y \partial x} = -\Omega \qquad \text{and} \qquad \frac{\partial^2 \phi}{\partial x \partial y} = \Omega.$$

This is not possible if $\Omega \neq 0$ because the smoothness of $\mathbf{v}$ implies that its potential should be smooth, so the mixed partials should be equal. Thus, no such ϕ can exist; $\mathbf{v}$ is not conservative. ■

BEWARE
Do not confuse this **necessary condition** with a **sufficient condition** to guarantee that **F** is conservative. We will show later that more than just $\partial F_1/\partial y = \partial F_2/\partial x$ on D is necessary to guarantee that **F** is conservative on D.

Example 2 suggests a condition that must be satisfied by any conservative plane vector field.

Necessary condition for a conservative plane vector field

If $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ is a conservative vector field in a domain D of the xy-plane, then the condition

$$\frac{\partial}{\partial y}F_1(x, y) = \frac{\partial}{\partial x}F_2(x, y)$$

must be satisfied at all points of D.

To see this, observe that

$$F_1\mathbf{i} + F_2\mathbf{j} = \mathbf{F} = \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j}$$

implies the two scalar equations

$$F_1 = \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad F_2 = \frac{\partial \phi}{\partial y},$$

and since the mixed partial derivatives of ϕ should be equal,

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial F_2}{\partial x}.$$

A similar condition obtains for vector fields in 3-space.

Necessary conditions for a conservative vector field in 3-space

If $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ is a conservative vector field in a domain D in 3-space, then we must have, everywhere in D,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

Equipotential Surfaces and Curves

If $\phi(x, y, z)$ is a potential function for the conservative vector field **F**, then the *level surfaces* $\phi(x, y, z) = C$ of ϕ are called **equipotential surfaces** of **F**. Since $\mathbf{F} = \nabla\phi$ is normal to these surfaces (wherever it does not vanish), the field lines of **F** always intersect the equipotential surfaces at right angles. For instance, the equipotential surfaces of the gravitational force field of a point mass are spheres centred at the point; these spheres are normal to the field lines, which are straight lines passing through the point. Similarly, for a conservative plane vector field, the *level curves* of the potential function are called **equipotential curves** of the vector field. They are the **orthogonal trajectories** of the field lines; that is, they intersect the field lines at right angles.

Example 3 Show that the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ is conservative and find a potential function for it. Describe the field lines and the equipotential curves.

Solution Since $\partial F_1/\partial y = 0 = \partial F_2/\partial x$ everywhere in $\mathbb{R}^2$, we would expect $\mathbf{F}$ to be conservative. Any potential function ϕ must satisfy

$$\frac{\partial \phi}{\partial x} = F_1 = x \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = F_2 = -y.$$

The first of these equations gives
get

$$-y = \frac{\partial \phi}{\partial y} = C_1{}'(y) \quad \Rightarrow \quad C_1(y) = -\frac{1}{2}y^2 + C_2.$$

Thus, $\mathbf{F}$ is conservative and, for any constant C_2,

$$\phi(x, y) = \frac{x^2 - y^2}{2} + C_2$$

is a potential function for $\mathbf{F}$. The field lines of $\mathbf{F}$ satisfy

$$\frac{dx}{x} = -\frac{dy}{y} \quad \Rightarrow \quad \ln|x| = -\ln|y| + \ln C_3 \quad \Rightarrow \quad xy = C_3.$$

The field lines of $\mathbf{F}$ are thus rectangular hyperbolas with the coordinate axes as asymptotes. The equipotential curves constitute another family of rectangular hyperbolas, $x^2 - y^2 = C_4$, with the lines $x = \pm y$ as asymptotes. Curves of the two families intersect at right angles. (See Figure 15.5.) Note, however, that $\mathbf{F}$ does not specify a direction at the origin and the orthogonality breaks down there; in fact, neither family has a unique curve through that point. ■

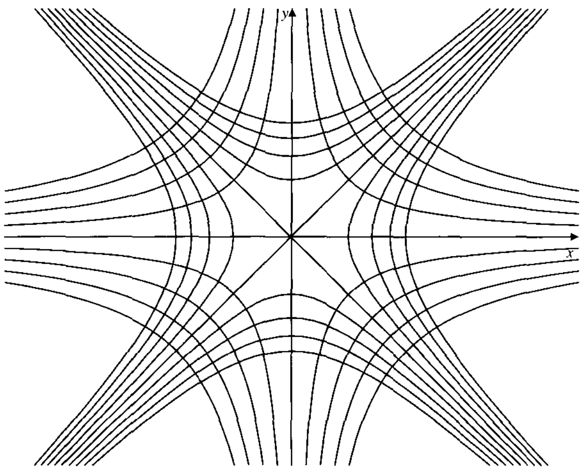

Figure 15.5 The field lines (black) and equipotential curves (colour) for the field $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$

Remark In the above example we constructed the potential ϕ by first integrating $\partial\phi/\partial x = F_1$. We could equally well have started by integrating $\partial\phi/\partial y = F_2$, in which case the constant of integration would have depended on x. In the end the same ϕ would have emerged.

Example 4 Decide whether the vector field

$$\mathbf{F} = \left(xy - \sin z\right)\mathbf{i} + \left(\frac{1}{2}x^2 - \frac{e^y}{z}\right)\mathbf{j} + \left(\frac{e^y}{z^2} - x\cos z\right)\mathbf{k}$$

is conservative in $D = \{(x, y, z) : z \neq 0\}$, and find a potential if it is.

Solution Note that $\mathbf{F}$ is not defined when $z = 0$. However, since

$$\frac{\partial F_1}{\partial y} = x = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = -\cos z = \frac{\partial F_3}{\partial x}, \quad \text{and} \quad \frac{\partial F_2}{\partial z} = \frac{e^y}{z^2} = \frac{\partial F_3}{\partial y},$$

$\mathbf{F}$ may still be conservative in domains not intersecting the xy-plane $z = 0$. If so, its potential ϕ should satisfy

$$\frac{\partial \phi}{\partial x} = xy - \sin z, \quad \frac{\partial \phi}{\partial y} = \frac{1}{2}x^2 - \frac{e^y}{z}, \quad \text{and} \quad \frac{\partial \phi}{\partial z} = \frac{e^y}{z^2} - x\cos z. \qquad (*)$$

From the first equation of $(*)$,

$$\phi(x, y, z) = \int (xy - \sin z)\,dx = \frac{1}{2}x^2 y - x\sin z + C_1(y, z).$$

(Again, note that the constant of integration can be a function of any parameters of the integrand; it is constant only with respect to the variable of integration.) Using the second equation of $(*)$, we obtain

$$\frac{1}{2}x^2 - \frac{e^y}{z} = \frac{\partial \phi}{\partial y} = \frac{1}{2}x^2 + \frac{\partial C_1(y, z)}{\partial y}.$$

Thus,

$$C_1(y, z) = -\int \frac{e^y}{z}\,dy = -\frac{e^y}{z} + C_2(z)$$

and

$$\phi(x, y, z) = \frac{1}{2}x^2 y - x\sin z - \frac{e^y}{z} + C_2(z).$$

Finally, using the third equation of $(*)$,

$$\frac{e^y}{z^2} - x\cos z = \frac{\partial \phi}{\partial z} = -x\cos z + \frac{e^y}{z^2} + C_2{}'(z).$$

Thus $C_2{}'(z) = 0$ and $C_2(z) = C$ (a constant). Indeed, $\mathbf{F}$ is conservative and, for any constant C,

$$\phi(x, y, z) = \frac{1}{2}x^2 y - x\sin z - \frac{e^y}{z} + C$$

is a potential function for $\mathbf{F}$ in the given domain D. C may have different values in the two regions $z > 0$ and $z < 0$ whose union constitutes D.

■

Remark If, in the above solution, the differential equation for $C_1(y, z)$ had involved x or if that for $C_2(z)$ had involved either x or y, we would not have been able to find ϕ. This did not happen because of the three conditions on the partials of F_1, F_2, and F_3 verified at the outset.

Remark The existence of a potential for a vector field depends on the *topology* of the domain of the field (i.e., whether the domain has *holes* in it and what kind of holes) as well as on the structure of the components of the field itself. (Even if the necessary conditions given above are satisfied, a vector field may not be conservative in a domain that has *holes*.) We will be probing further into the nature of conservative vector fields in Section 15.4 and in the next chapter; we will eventually show that the above *necessary conditions* are also *sufficient* to guarantee that **F** is conservative if the domain of **F** satisfies certain conditions. At this point, however, we give an example in which a plane vector field fails to be conservative on a domain where the necessary condition is, nevertheless, satisfied.

Example 5 For $(x, y) \neq (0, 0)$, define a vector field $\mathbf{F}(x, y)$ and a scalar field $\theta(x, y)$ as follows:

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}\right)\mathbf{i} + \left(\frac{x}{x^2 + y^2}\right)\mathbf{j}$$

$$\theta(x, y) = \text{the polar angle } \theta \text{ of } (x, y) \text{ such that } 0 \leq \theta < 2\pi.$$

Thus, $x = r\cos\theta(x, y)$ and $y = r\sin\theta(x, y)$, where $r^2 = x^2 + y^2$. Verify the following:

(a) $\dfrac{\partial}{\partial y}F_1(x, y) = \dfrac{\partial}{\partial x}F_2(x, y)$ for $(x, y) \neq (0, 0)$.

(b) $\nabla\theta(x, y) = \mathbf{F}(x, y)$ for all $(x, y) \neq (0, 0)$ such that $0 < \theta < 2\pi$.

(c) **F** is not conservative on the whole xy-plane excluding the origin.

Solution

(a) We have $F_1 = \dfrac{-y}{x^2 + y^2}$ and $F_2 = \dfrac{x}{x^2 + y^2}$. Thus

$$\begin{aligned}\frac{\partial}{\partial y}F_1(x, y) &= \frac{\partial}{\partial y}\left(-\frac{y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right)\\ &= \frac{\partial}{\partial x}F_2(x, y)\end{aligned}$$

for all $(x, y) \neq (0, 0)$.

(b) Differentiate the equations $x = r\cos\theta$ and $y = r\sin\theta$ implicitly with respect to x to obtain

$$\begin{aligned}1 &= \frac{\partial x}{\partial x} = \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\frac{\partial\theta}{\partial x},\\ 0 &= \frac{\partial y}{\partial x} = \frac{\partial r}{\partial x}\sin\theta + r\cos\theta\frac{\partial\theta}{\partial x}.\end{aligned}$$

Eliminating $\partial r/\partial x$ from this pair of equations and solving for $\partial\theta/\partial x$ leads to

$$\frac{\partial\theta}{\partial x} = -\frac{r\sin\theta}{r^2} = -\frac{y}{x^2 + y^2} = F_1.$$

Similarly, differentiation with respect to y produces

$$\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} = F_2.$$

These formulas hold only if $0 < \theta < 2\pi$; θ is not even continuous on the positive x-axis; if $x > 0$, then

$$\lim_{y\to 0+} \theta(x, y) = 0 \qquad \text{but} \qquad \lim_{y\to 0-} \theta(x, y) = 2\pi.$$

Thus, $\nabla\theta = \mathbf{F}$ holds everywhere in the plane except at points $(x, 0)$ where $x \geq 0$.

(c) Suppose that $\mathbf{F}$ is conservative on the whole plane excluding the origin. Then $\mathbf{F} = \nabla\phi$ there, for some scalar function $\phi(x, y)$. Then $\nabla(\theta - \phi) = \mathbf{0}$ for $0 < \theta < 2\pi$, and $\theta - \phi = C$ (constant), or $\theta = \phi + C$. The left side of this equation is discontinuous along the positive x-axis but the right side is not. Therefore the two sides cannot be equal. This contradiction shows that $\mathbf{F}$ cannot be conservative on the whole plane excluding the origin. ■

Remark Observe that the origin $(0, 0)$ is a *hole* in the domain of $\mathbf{F}$ in the above example. While $\mathbf{F}$ satisfies the necessary condition for being conservative everywhere except at this hole, you must remove from the domain of $\mathbf{F}$ a half-line (ray), or, more generally, a curve from the origin to infinity in order to get a potential function for $\mathbf{F}$. $\mathbf{F}$ is *not* conservative on any domain containing a curve that surrounds the origin. Exercises 22–24 of Section 15.4 will shed further light on this situation.

Sources, Sinks, and Dipoles

Imagine that 3-space is filled with an incompressible fluid emitted by a point source at the origin at a volume rate $dV/dt = 4\pi m$. (We say that the origin is a **source** of strength m.) By symmetry, the fluid flows outward on radial lines from the origin with equal speed at equal distances from the origin in all directions, and the fluid emitted at the origin at some instant $t = 0$ will at later time t be spread over a spherical surface of radius $r = r(t)$. All the fluid inside that sphere was emitted in the time interval $[0, t]$, so we have

$$\frac{4}{3}\pi r^3 = 4\pi m t.$$

Differentiating this equation with respect to t we obtain $r^2(dr/dt) = m$, and the outward speed of the fluid at distance r from the origin is $v(r) = m/r^2$. The velocity field of the moving fluid is therefore

$$\mathbf{v}(\mathbf{r}) = v(r)\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{m}{r^3}\mathbf{r}.$$

This velocity field is conservative (except at the origin) and has potential

$$\phi(\mathbf{r}) = -\frac{m}{r}.$$

A **sink** is a negative source. A sink of strength m at the origin (which annihilates or sucks up fluid at a rate $dV/dt = 4\pi m$) has velocity field and potential given by

$$\mathbf{v}(\mathbf{r}) = -\frac{m}{r^3}\mathbf{r} \qquad \text{and} \qquad \phi(\mathbf{r}) = \frac{m}{r}.$$

The potentials or velocity fields of sources or sinks located at other points are obtained by translation of these formulas; for instance, the velocity field of a source of strength m at the point with position vector $\mathbf{r}_0$ is

$$\mathbf{v}(\mathbf{r}) = -\nabla\left(\frac{m}{|\mathbf{r}-\mathbf{r}_0|}\right) = \frac{m}{|\mathbf{r}-\mathbf{r}_0|^3}(\mathbf{r}-\mathbf{r}_0).$$

This should be compared with the gravitational force field due to a mass m at the origin. The two are the same except for sign and a constant related to units of measurement. For this reason we regard a point mass as a sink for its own gravitational field. Similarly, the electrostatic field due to a point charge q at $\mathbf{r}_0$

In general, the field lines of a vector field converge at a source or sink of that field.

A **dipole** is a system consisting of a source and a sink of equal strength m separated by a short distance ℓ. The product $\mu = m\ell$ is called the **dipole moment**, and the line containing the source and sink is called the **axis** of the dipole. Real physical dipoles, such as magnets, are frequently modelled by *ideal* dipoles that are the limits of such real dipoles as $m \to \infty$ and $\ell \to 0$ in such a way that the dipole moment μ remains constant.

Example 6 Calculate the velocity field, $\mathbf{v}(x, y, z)$, associated with a dipole of moment μ located at the origin and having axis along the z-axis.

Solution We start with a source of strength m at position $(0, 0, \ell/2)$ and a sink of strength m at $(0, 0, -\ell/2)$. The potential of this system is

$$\phi(\mathbf{r}) = -m\left(\frac{1}{|\mathbf{r}-\frac{1}{2}\ell\mathbf{k}|} - \frac{1}{|\mathbf{r}+\frac{1}{2}\ell\mathbf{k}|}\right).$$

The potential of the ideal dipole is the limit of the potential of this system as $m \to \infty$ and $\ell \to 0$ in such a way that $m\ell = \mu$:

$$\begin{aligned}\phi(\mathbf{r}) &= \lim_{\substack{\ell\to 0\\ ml=\mu}} -m\left(\frac{|\mathbf{r}+\frac{1}{2}\ell\mathbf{k}| - |\mathbf{r}-\frac{1}{2}\ell\mathbf{k}|}{|\mathbf{r}+\frac{1}{2}\ell\mathbf{k}|\,|\mathbf{r}-\frac{1}{2}\ell\mathbf{k}|}\right)\\ &= -\frac{\mu}{|\mathbf{r}|^2}\lim_{\ell\to 0}\frac{|\mathbf{r}+\frac{1}{2}\ell\mathbf{k}| - |\mathbf{r}-\frac{1}{2}\ell\mathbf{k}|}{\ell}\end{aligned}$$

(now use l'Hôpital's Rule and the rule for differentiating lengths of vectors)

$$\begin{aligned}&= -\frac{\mu}{|\mathbf{r}|^2}\lim_{\ell\to 0}\frac{\dfrac{(\mathbf{r}+\frac{1}{2}\ell\mathbf{k})\bullet\frac{1}{2}\mathbf{k}}{|\mathbf{r}+\frac{1}{2}\ell\mathbf{k}|} - \dfrac{(\mathbf{r}-\frac{1}{2}\ell\mathbf{k})\bullet(-\frac{1}{2}\mathbf{k})}{|\mathbf{r}-\frac{1}{2}\ell\mathbf{k}|}}{1}\\ &= -\frac{\mu}{|\mathbf{r}|^2}\lim_{\ell\to 0}\left(\frac{\frac{1}{2}z+\frac{1}{4}\ell}{|\mathbf{r}+\frac{1}{2}\ell\mathbf{k}|} + \frac{\frac{1}{2}z-\frac{1}{4}\ell}{|\mathbf{r}-\frac{1}{2}\ell\mathbf{k}|}\right)\\ &= -\frac{\mu z}{|\mathbf{r}|^3}.\end{aligned}$$

The required velocity field is the gradient of this potential. We have

$$\frac{\partial \phi}{\partial x} = \frac{3\,\mu z}{|\mathbf{r}|^4}\frac{\mathbf{r}\bullet\mathbf{i}}{|\mathbf{r}|} = \frac{3\mu xz}{|\mathbf{r}|^5}$$

$$\frac{\partial \phi}{\partial y} = \frac{3\mu yz}{|\mathbf{r}|^5}$$

$$\frac{\partial \phi}{\partial z} = -\frac{\mu}{|\mathbf{r}|^3} + \frac{3\mu z^2}{|\mathbf{r}|^5} = \frac{\mu(2z^2 - x^2 - y^2)}{|\mathbf{r}|^5}$$

$$\mathbf{v}(\mathbf{r}) = \nabla\phi(\mathbf{r}) = \frac{\mu}{|\mathbf{r}|^5}\big(3xz\mathbf{i} + 3yz\mathbf{j} + (2z^2 - x^2 - y^2)\mathbf{k}\big).$$

Some streamlines for a plane cross-section containing the z-axis are shown in Figure 15.6.

■

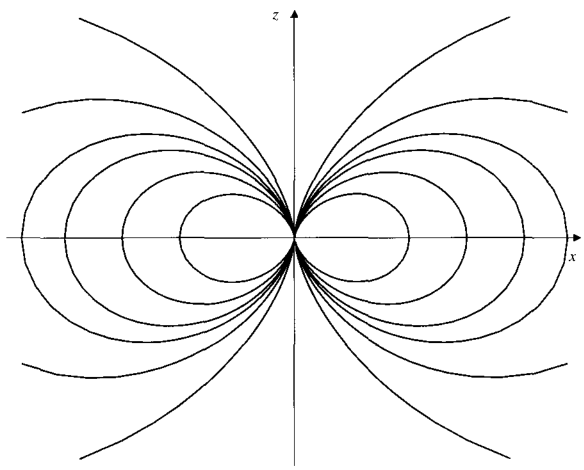

Figure 15.6 Streamlines of a dipole

Exercises 15.2

In Exercises 1–6, determine whether the given vector field is conservative, and find a potential if it is.

1. $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + 3z\mathbf{k}$
2. $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$
3. $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} - y\mathbf{j}}{x^2 + y^2}$ 4. $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$
5. $\mathbf{F}(x, y, z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$
6. $\mathbf{F}(x, y, z) = e^{x^2+y^2+z^2}(xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k})$
7. Find the three-dimensional vector field with potential $\phi(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}_0|^2}$.
8. Calculate $\nabla \ln |\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
9. Show that the vector field
$$\mathbf{F}(x, y, z) = \frac{2x}{z}\mathbf{i} + \frac{2y}{z}\mathbf{j} - \frac{x^2 + y^2}{z^2}\mathbf{k}$$
is conservative, and find its potential. Describe the equipotential surfaces. Find the field lines of $\mathbf{F}$.
10. Repeat Exercise 9 for the field
$$\mathbf{F}(x, y, z) = \frac{2x}{z}\mathbf{i} + \frac{2y}{z}\mathbf{j} + \left(1 - \frac{x^2 + y^2}{z^2}\right)\mathbf{k}.$$
11. Find the velocity field due to two sources of strength m, one located at $(0, 0, \ell)$ and the other at $(0, 0, -\ell)$. Where is the velocity zero? Find the velocity at any point $(x, y, 0)$ in the xy-plane. Where in the xy-plane is the speed greatest?

* 12. Find the velocity field for a system consisting of a source of strength 2 at the origin and a sink of strength 1 at $(0, 0, 1)$. Show that the velocity is vertical at all points of a certain sphere. Sketch the streamlines of the flow.

Exercises 13–18 provide an analysis of two-dimensional sources and dipoles similar to that developed for three dimensions in the text.

13. In 3-space filled with an incompressible fluid, we say that the z-axis is a **line source** of strength m if every interval Δz

along that axis emits fluid at volume rate $dV/dt = 2\pi m \Delta z$. The fluid then spreads out symmetrically in all directions perpendicular to the z-axis. Show that the velocity field of the flow is

$$\mathbf{v} = \frac{m}{x^2 + y^2}(x\mathbf{i} + y\mathbf{j}).$$

14. The flow in Exercise 13 is two-dimensional because $\mathbf{v}$ depends only on x and y and has no component in the z direction. Regarded as a *plane* vector field, it is the field of a two-dimensional point source of strength m located at the origin (i.e., fluid is emitted at the origin at the *areal rate* $dA/dt = 2\pi m$). Show that the vector field is conservative, and find a potential function $\phi(x, y)$ for it.

* **15.** Find the potential, ϕ, and the field, $\mathbf{F} = \nabla\phi$, for a two-dimensional dipole at the origin, with axis in the y direction and dipole moment μ. Such a dipole is the limit of a system consisting of a source of strength m at $(0, \ell/2)$ and a sink of strength m at $(0, -\ell/2)$, as $\ell \to 0$ and $m \to \infty$ such that $m\ell = \mu$.

* **16.** Show that the equipotential curves of the two-dimensional dipole in Exercise 15 are circles tangent to the x-axis at the origin.

* **17.** Show that the streamlines (field lines) of the two-dimensional dipole in Exercises 15 and 16 are circles tangent to the y-axis at the origin. *Hint:* it is possible to do this geometrically. If you choose to do it by setting up a differential equation, you may find the change of dependent variable

$$y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

useful for integrating the equation.

* **18.** Show that the velocity field of a line source of strength $2m$ can be found by integrating the (three-dimensional) velocity field of a point source of strength $m\,dz$ at $(0, 0, z)$ over the whole z-axis. Why does the integral correspond to a line source of strength $2m$ rather than strength m? Can the potential of the line source be obtained by integrating the potentials of the point sources?

19. Show that the gradient of a function expressed in terms of polar coordinates in the plane is

$$\nabla\phi(r, \theta) = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}}.$$

(This is a repeat of Exercise 16 in Section 12.7.)

20. Use the result of Exercise 19 to show that a necessary condition for the vector field

$$\mathbf{F}(r, \theta) = F_r(r, \theta)\hat{\mathbf{r}} + F_\theta(r, \theta)\hat{\boldsymbol{\theta}}$$

(expressed in terms of polar coordinates) to be conservative is that

$$\frac{\partial F_r}{\partial\theta} - r\frac{\partial F_\theta}{\partial r} = F_\theta.$$

21. Show that $\mathbf{F} = r\sin 2\theta\hat{\mathbf{r}} + r\cos 2\theta\hat{\boldsymbol{\theta}}$ is conservative, and find a potential for it.

22. For what values of the constants α and β is the vector field

$$\mathbf{F} = r^2\cos\theta\hat{\mathbf{r}} + \alpha r^\beta \sin\theta\hat{\boldsymbol{\theta}}$$

conservative? Find a potential for $\mathbf{F}$ if α and β have these values.

15.3 Line Integrals

If a wire stretched out along the x-axis from $x = a$ to $x = b$ has constant *line density* δ (units of mass per unit length), then the total mass of the wire will be $m = \delta(b - a)$. If the density of the wire is not constant but varies continuously from point to point (e.g., because the thickness of the wire is not uniform), then we must find the total mass of the wire by "summing" (i.e., integrating) differential elements of mass $dm = \delta(x)\,dx$:

$$m = \int_a^b \delta(x)\,dx.$$

In general, the definite integral, $\int_a^b f(x)\,dx$, represents the *total amount* of a quantity distributed along the x-axis between a and b in terms of the *line density*, $f(x)$, of that quantity at point x. The amount of the quantity in an *infinitesimal* interval of length dx at x is $f(x)\,dx$, and the integral adds up these infinitesimal contributions (or *elements*) to give the total amount of the quantity. Similarly, the integrals $\iint_D f(x, y)\,dA$ and $\iiint_R f(x, y, z)\,dV$ represent the total amounts of quantities distributed over regions D in the plane and R in 3-space in terms of the *areal* or *volume* densities of these quantities.

It may happen that a quantity is distributed with specified line density along a *curve* in the plane or in 3-space, or with specified areal density over a *surface* in 3-space. In such cases we require *line integrals* or *surface integrals* to add up the contributing elements and calculate the total quantity. We examine line integrals in this section and the next and surface integrals in Sections 15.5 and 15.6.

Let $\mathcal{C}$ be a bounded, continuous parametric curve in $\mathbb{R}^3$. Recall (from Section 11.1) that $\mathcal{C}$ is a *smooth curve* if it has a parametrization of the form

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad t \quad \text{in interval} \quad I,$$

with "velocity" vector $\mathbf{v} = d\mathbf{r}/dt$ continuous and nonzero. We will call $\mathcal{C}$ a **smooth arc** if it is a smooth curve with *finite* parameter interval $I = [a, b]$.

In Section 11.3 we saw how to calculate the length of $\mathcal{C}$ by subdividing it into short arcs using points corresponding to parameter values

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b,$$

adding up the lengths $|\Delta\mathbf{r}_i| = |\mathbf{r}_i - \mathbf{r}_{i-1}|$ of line segments joining these points, and taking the limit as the maximum distance between adjacent points approached zero. The length was denoted

$$\int_{\mathcal{C}} ds$$

and is a special example of a line integral along $\mathcal{C}$ having integrand 1.

The line integral of a general function $f(x, y, z)$ can be defined similarly. We choose a point (x_i^*, y_i^*, z_i^*) on the ith subarc and form the Riemann sum

$$S_n = \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,|\Delta\mathbf{r}_i|.$$

If this sum has a limit as $\max |\Delta\mathbf{r}_i| \to 0$, independent of the particular choices of the points (x_i^*, y_i^*, z_i^*), then we call this limit the **line integral** of f along $\mathcal{C}$ and denote it by

$$\int_{\mathcal{C}} f(x, y, z)\,ds.$$

If $\mathcal{C}$ is a smooth arc and if f is continuous on $\mathcal{C}$, then the limit will certainly exist; its value is given by a definite integral of a continuous function, as shown in the next paragraph. It will also exist (for continuous f) if $\mathcal{C}$ is **piecewise smooth**, consisting of finitely many smooth arcs linked end to end; in this case the line integral of f along $\mathcal{C}$ is the sum of the line integrals of f along each of the smooth arcs. Improper line integrals can also be considered, where f has discontinuities or where the length of a curve is not finite.

Evaluating Line Integrals

The length of $\mathcal{C}$ was evaluated by expressing the arc length element $ds = |d\mathbf{r}/dt|\,dt$ in terms of a parametrization $\mathbf{r} = \mathbf{r}(t)$, $(a \le t \le b)$ of the curve, and integrating this from $t = a$ to $t = b$:

$$\text{length of } \mathcal{C} = \int_{\mathcal{C}} ds = \int_a^b \left|\frac{d\mathbf{r}}{dt}\right| dt.$$

More general line integrals are evaluated similarly:

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| dt.$$

Of course, all of the above discussion applies equally well to line integrals of functions $f(x, y)$ along curves $\mathcal{C}$ in the xy-plane.

Remark It should be noted that the value of the line integral of a function f along a curve $\mathcal{C}$ depends on f and $\mathcal{C}$ but not on the particular way $\mathcal{C}$ is parametrized. If $\mathbf{r} = \mathbf{r}^*(u)$, $\alpha \le u \le \beta$, is another parametrization of the same smooth curve $\mathcal{C}$, then any point $\mathbf{r}(t)$ on $\mathcal{C}$ can be expressed in terms of the new parametrization as $\mathbf{r}^*(u)$, where u depends on t: $u = u(t)$. If $\mathbf{r}^*(u)$ traces $\mathcal{C}$ in the same direction as $\mathbf{r}(t)$, then $u(a) = \alpha$, $u(b) = \beta$, and $du/dt \ge 0$; if $\mathbf{r}^*(u)$ traces $\mathcal{C}$ in the opposite direction, then $u(a) = \beta$, $u(b) = \alpha$, and $du/dt \le 0$. In either event,

$$\int_a^b f\big(\mathbf{r}(t)\big) \left|\frac{d\mathbf{r}}{dt}\right| dt = \int_a^b f\big(\mathbf{r}^*(u(t))\big) \left|\frac{d\mathbf{r}^*}{du}\frac{du}{dt}\right| dt = \int_\alpha^\beta f\big(\mathbf{r}^*(u)\big) \left|\frac{d\mathbf{r}^*}{du}\right| du.$$

Thus, the line integral is *independent of parametrization* of the curve $\mathcal{C}$. The following example illustrates this fact.

Example 1 A circle of radius $a > 0$ has centre at the origin in the xy-plane. Let $\mathcal{C}$ be the half of this circle lying in the half-plane $y \ge 0$. Use two different parametrizations of $\mathcal{C}$ to find the moment of $\mathcal{C}$ about $y = 0$.

Solution We are asked to calculate $\displaystyle\int_{\mathcal{C}} y\, ds$.

$\mathcal{C}$ can be parametrized $\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $(0 \le t \le \pi)$. Therefore,

$$\frac{d\mathbf{r}}{dt} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} \qquad \text{and} \qquad \left|\frac{d\mathbf{r}}{dt}\right| = a,$$

and the moment of $\mathcal{C}$ about $y = 0$ is

$$\int_{\mathcal{C}} y\, ds = \int_0^\pi a\sin t\, a\, dt = -a^2\cos t\Big|_0^\pi = 2a^2.$$

$\mathcal{C}$ can also be parametrized $\mathbf{r} = x\,\mathbf{i} + \sqrt{a^2 - x^2}\,\mathbf{j}$, $(-a \le x \le a)$, for which we have

$$\frac{d\mathbf{r}}{dx} = \mathbf{i} - \frac{x}{\sqrt{a^2 - x^2}}\mathbf{j},$$
$$\left|\frac{d\mathbf{r}}{dx}\right| = \sqrt{1 + \frac{x^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}.$$

Thus, the moment of $\mathcal{C}$ about $y = 0$ is

$$\int_{\mathcal{C}} y\,ds = \int_{-a}^{a} \sqrt{a^2 - x^2}\,\frac{a}{\sqrt{a^2 - x^2}}\,dx = a\int_{-a}^{a} dx = 2a^2.$$

It is comforting to get the same answer using different parametrizations. Unlike the line integrals of vector fields considered in the next section, the line integrals of scalar fields considered here do not depend on the direction (orientation) of $\mathcal{C}$. The two parametrizations of the semicircle were in opposite directions but still gave the same result.

Line integrals frequently lead to definite integrals that are very difficult or impossible to evaluate without using numerical techniques. Only very simple curves and ones that have been contrived to lead to simple expressions for ds are amenable to exact calculation of line integrals.

Example 2 Find the centroid of the circular helix $\mathcal{C}$ given by

$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

Solution As we observed in Example 5 of Section 11.3, for this helix $ds = \sqrt{a^2 + b^2}\,dt$. On the helix we have $z = bt$, so its moment about $z = 0$ is

$$M_{z=0} = \int_{\mathcal{C}} z\,ds = b\sqrt{a^2 + b^2}\int_0^{2\pi} t\,dt = 2\pi^2 b\sqrt{a^2 + b^2}.$$

Since the helix has length $L = 2\pi\sqrt{a^2 + b^2}$, the z-component of its centroid is $M_{z=0}/L = \pi b$. The moment of the helix about $x = 0$ is

$$M_{x=0} = \int_{\mathcal{C}} x\,ds = a\sqrt{a^2 + b^2}\int_0^{2\pi} \cos t\,dt = 0,$$
$$M_{y=0} = \int_{\mathcal{C}} y\,ds = a\sqrt{a^2 + b^2}\int_0^{2\pi} \sin t\,dt = 0.$$

Thus the centroid is $(0, 0, \pi b)$.

Sometimes a curve, along which a line integral is to be taken, is specified as the intersection of two surfaces with given equations. It is normally necessary to parametrize the curve in order to evaluate a line integral. Recall from Section 11.3 that if one of the surfaces is a cylinder parallel to one of the coordinate axes, it is usually easiest to begin by parametrizing that cylinder. (Otherwise, combine the equations to eliminate one variable and thus obtain such a cylinder on which the curve lies.)

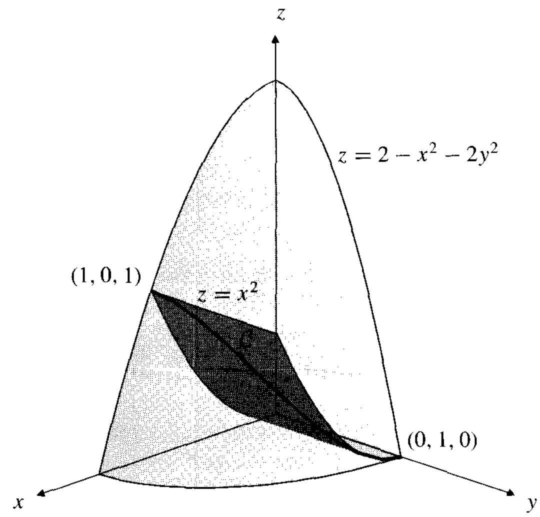

Figure 15.7 The curve of intersection of $z = x^2$ and $z = 2 - x^2 - 2y^2$

Example 3 Find the mass of a wire lying along the first octant part $\mathcal{C}$ of the curve of intersection of the elliptic paraboloid $z = 2 - x^2 - 2y^2$ and the parabolic cylinder $z = x^2$ between $(0, 1, 0)$ and $(1, 0, 1)$ (see Figure 15.7) if the density of the wire at position (x, y, z) is $\delta(x, y, z) = xy$.

Solution We need a convenient parametrization of $\mathcal{C}$. Since the curve $\mathcal{C}$ lies on the cylinder $z = x^2$ and x goes from 0 to 1, we can let $x = t$ and $z = t^2$. Thus, $2y^2 = 2 - x^2 - z = 2 - 2t^2$, so $y^2 = 1 - t^2$. Since $\mathcal{C}$ lies in the first octant, it can be parametrized by

$$x = t, \qquad y = \sqrt{1 - t^2}, \qquad z = t^2, \qquad (0 \le t \le 1).$$

Then $dx/dt = 1$, $dy/dt = -t/\sqrt{1 - t^2}$, and $dz/dt = 2t$, so

$$ds = \sqrt{1 + \frac{t^2}{1 - t^2} + 4t^2}\, dt = \frac{\sqrt{1 + 4t^2 - 4t^4}}{\sqrt{1 - t^2}}\, dt.$$

Hence, the mass of the wire is

$$\begin{aligned} m &= \int_{\mathcal{C}} xy\, ds = \int_0^1 t\sqrt{1 - t^2}\frac{\sqrt{1 + 4t^2 - 4t^4}}{\sqrt{1 - t^2}}\, dt \\ &= \int_0^1 t\sqrt{1 + 4t^2 - 4t^4}\, dt \qquad \text{Let } u = t^2.. \\ &= \frac{1}{2}\int_0^1 \sqrt{1 + 4u - 4u^2}\, du \\ &= \frac{1}{2}\int_0^1 \sqrt{2 - (2u - 1)^2}\, du \qquad \text{Let } v = 2u - 1.. \\ &= \frac{1}{4}\int_{-1}^1 \sqrt{2 - v^2}\, dv = \frac{1}{2}\int_0^1 \sqrt{2 - v^2}\, dv \\ &= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \frac{\pi + 2}{8}. \end{aligned}$$

(The final integral above was evaluated by interpreting it as the area of part of a circle. You are invited to supply the details. It can also be done by the substitution $v = \sqrt{2}\sin w$.)

Exercises 15.3

1. Show that the curve $\mathcal{C}$ given by

$$\mathbf{r} = a\cos t\sin t\,\mathbf{i} + a\sin^2 t\,\mathbf{j} + a\cos t\,\mathbf{k}, \quad (0 \le t \le \tfrac{\pi}{2}),$$

lies on a sphere centred at the origin. Find $\int_{\mathcal{C}} z\,ds$.

2. Let $\mathcal{C}$ be the conical helix with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$, $(0 \le t \le 2\pi)$. Find $\int_{\mathcal{C}} z\,ds$.

3. Find the mass of a wire along the curve

$$\mathbf{r} = 3t\mathbf{i} + 3t^2\mathbf{j} + 2t^3\mathbf{k}, \quad (0 \le t \le 1),$$

if the density at $\mathbf{r}(t)$ is $1 + t$ g/unit length.

4. Show that the curve $\mathcal{C}$ in Example 3 also has parametrization $x = \cos t$, $y = \sin t$, $z = \cos^2 t$, $(0 \le t \le \pi/2)$, and recalculate the mass of the wire in that example using this parametrization.

5. Find the moment of inertia about the z-axis (i.e., the value of $\delta\int_{\mathcal{C}}(x^2 + y^2)\,ds$), for a wire of constant density δ lying along the curve $\mathcal{C}$: $\mathbf{r} = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + t\mathbf{k}$, from $t = 0$ to $t = 2\pi$.

6. Evaluate $\int_{\mathcal{C}} e^z\,ds$, where $\mathcal{C}$ is the curve in Exercise 5.

7. Find $\int_{\mathcal{C}} x^2\,ds$ along the line of intersection of the two planes $x - y + z = 0$, and $x + y + 2z = 0$, from the origin to the point $(3, 1, -2)$.

8. Find $\int_{\mathcal{C}} \sqrt{1 + 4x^2z^2}\,ds$, where $\mathcal{C}$ is the curve of intersection of the surfaces $x^2 + z^2 = 1$ and $y = x^2$.

9. Find the mass and centre of mass of a wire bent in the shape of the circular helix $x = \cos t$, $y = \sin t$, $z = t$, $(0 \le t \le 2\pi)$ if the wire has line density given by $\delta(x, y, z) = z$.

10. Repeat Exercise 9 for the part of the wire corresponding to $0 \le t \le \pi$.

11. Find the moment of inertia about the y-axis, that is,

$$\int_{\mathcal{C}}(x^2 + z^2)\,ds,$$

of the curve $x = e^t$, $y = \sqrt{2}\,t$, $z = e^{-t}$, $(0 \le t \le 1)$.

12. Find the centroid of the curve in Exercise 11.

* **13.** Find $\int_{\mathcal{C}} x\,ds$ along the first octant part of the curve of intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $z = x$.

* **14.** Find $\int_{\mathcal{C}} z\,ds$ along the part of the curve $x^2 + y^2 + z^2 = 1$, $x + y = 1$, where $z \ge 0$.

* **15.** Find $\int_{\mathcal{C}} \dfrac{ds}{(2y^2 + 1)^{3/2}}$, where $\mathcal{C}$ is the parabola $z^2 = x^2 + y^2$, $x + z = 1$. *Hint:* use $y = t$ as parameter.

16. Express as a definite integral, but do not try to evaluate, the value of $\int_{\mathcal{C}} xyz\,ds$, where $\mathcal{C}$ is the curve $y = x^2$, $z = y^2$ from $(0, 0, 0)$ to $(2, 4, 16)$.

* **17.** The function

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2\sin^2 t}\,dt$$

is called the **elliptic integral function of the second kind**. The **complete elliptic integral** of the second kind is the function $E(k) = E(k, \pi/2)$. In terms of these functions, express the length of one complete revolution of the elliptic helix

$$x = a\cos t, \qquad y = b\sin t, \qquad z = ct,$$

where $0 < a < b$. What is the length of that part of the helix lying between $t = 0$ and $t = T$, where $0 < T < \pi/2$?

* **18.** Evaluate $\int_{\mathcal{L}} \dfrac{ds}{x^2 + y^2}$, where $\mathcal{L}$ is the entire straight line with equation $Ax + By = C$, $(C \ne 0)$. *Hint:* use the symmetry of the integrand to replace the line with a line having a simpler equation but giving the same value to the integral.

15.4 Line Integrals of Vector Fields

In elementary physics the **work** done by a constant force of magnitude F in moving an object a distance d is defined to be the product of F and d: $W = Fd$. There is, however, a catch to this; it is understood that the force is exerted in the direction of motion of the object. If the object moves in a direction different from that of the force (because of some other forces acting on it), then the work done by the particular force is the product of the distance moved and the component of the force in the direction of motion. For instance, the work done by gravity in causing a 10 kg crate to slide 5 m down a ramp inclined at 45° to the horizontal is $W = 50g/\sqrt{2}$ N·m (where $g = 9.8$ m/s^2), since the scalar projection of the $10g$ N gravitational force on the crate in the direction of the ramp is $10g/\sqrt{2}$ N.

The work done by a *variable* force $\mathbf{F}(x, y, z) = \mathbf{F}(\mathbf{r})$, which depends continuously on position, in moving an object along a smooth curve $\mathcal{C}$ is the integral of *work elements* dW. The element dW corresponding to arc length element ds at position $\mathbf{r}$ on $\mathcal{C}$ is ds times the tangential component of the force $\mathbf{F}(\mathbf{r})$ along $\mathcal{C}$ in the direction of motion (see Figure 15.8):

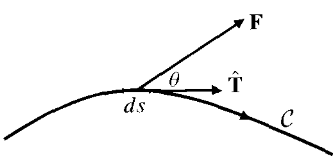

Figure 15.8 $dW = |\mathbf{F}|\cos\theta\, ds$
$= \mathbf{F} \bullet \hat{\mathbf{T}}\, ds$

$$dW = \mathbf{F}(\mathbf{r}) \bullet \hat{\mathbf{T}}\, ds.$$

Thus, the total work done by $\mathbf{F}$ in moving the object along $\mathcal{C}$ is

$$W = \int_{\mathcal{C}} \mathbf{F} \bullet \hat{\mathbf{T}}\, ds = \int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r},$$

where we use the vector differential $d\mathbf{r}$ as a convenient shorthand for $\hat{\mathbf{T}}\, ds$. Since

$$\hat{\mathbf{T}} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k},$$

we have

$$d\mathbf{r} = \hat{\mathbf{T}}\, ds = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k},$$

and the work W can be written in terms of the components of $\mathbf{F}$ as

$$W = \int_{\mathcal{C}} F_1\, dx + F_2\, dy + F_3\, dz.$$

In general, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a continuous vector field, and $\mathcal{C}$ is an oriented smooth curve, then the **line integral of the tangential component of F** along $\mathcal{C}$ is

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} &= \int_{\mathcal{C}} \mathbf{F} \bullet \hat{\mathbf{T}}\, ds \\ &= \int_{\mathcal{C}} F_1(x, y, z)\, dx + F_2(x, y, z)\, dy + F_3(x, y, z)\, dz.\end{aligned}$$

Such a line integral is sometimes called, somewhat improperly, the line integral of $\mathbf{F}$ along $\mathcal{C}$. (It is not the line integral of $\mathbf{F}$, which should have a vector value, but rather the line integral of the *tangential component* of $\mathbf{F}$, which has a scalar value.) Unlike the line integral considered in the previous section, this line integral depends on the direction of the orientation of $\mathcal{C}$; reversing the direction of $\mathcal{C}$ causes this line integral to change sign.

If $\mathcal{C}$ is a closed curve, the line integral of the tangential component of **F** around $\mathcal{C}$ is also called the **circulation** of **F** around $\mathcal{C}$. The fact that the curve is closed is often indicated by a small circle drawn on the integral sign;

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$$

denotes the circulation of **F** around the closed curve $\mathcal{C}$.

Like the line integrals studied in the previous section, a line integral of a continuous vector field is converted into an ordinary definite integral by using a parametrization of the path of integration. For a smooth arc

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad (a \le t \le b),$$

we have

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} &= \int_a^b \mathbf{F} \bullet \frac{d\mathbf{r}}{dt}\, dt \\ &= \int_a^b \left[F_1\big(x(t), y(t), z(t)\big)\frac{dx}{dt} + F_2\big(x(t), y(t), z(t)\big)\frac{dy}{dt} \right. \\ &\qquad \left. + F_3\big(x(t), y(t), z(t)\big)\frac{dz}{dt}\right] dt.\end{aligned}$$

Although this type of line integral changes sign if the orientation of $\mathcal{C}$ is reversed, it is otherwise independent of the particular parametrization used for $\mathcal{C}$. Again, a line integral over a piecewise smooth path is the sum of the line integrals over the individual smooth arcs constituting that path.

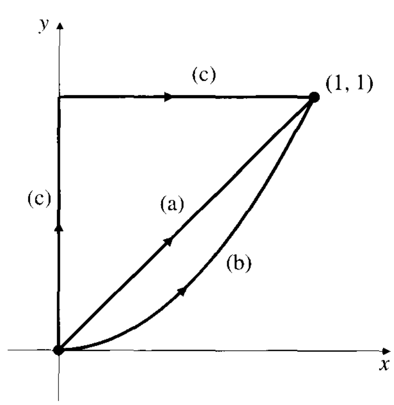

Figure 15.9 Three paths from (0, 0) to (1, 1)

Example 1 Let $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$. Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$$

from (0, 0) to (1, 1) along

(a) the straight line $y = x$,

(b) the curve $y = x^2$, and

(c) the piecewise smooth path consisting of the straight line segments from (0, 0) to (0, 1) and from (0, 1) to (1, 1).

Solution The three paths are shown in Figure 15.9. The straight path (a) can be parametrized $\mathbf{r} = t\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$. Thus $d\mathbf{r} = dt\mathbf{i} + dt\mathbf{j}$ and

$$\mathbf{F} \bullet d\mathbf{r} = (t^2\mathbf{i} + 2t^2\mathbf{j}) \bullet (\mathbf{i} + \mathbf{j})dt = 3t^2\, dt.$$

Therefore,

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_0^1 3t^2\, dt = t^3\Big|_0^1 = 1.$$

The parabolic path (b) can be parametrized $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$, so that $d\mathbf{r} = dt\mathbf{i} + 2t\,dt\mathbf{j}$. Thus,

$$\mathbf{F} \bullet d\mathbf{r} = (t^4\mathbf{i} + 2t^3\mathbf{j}) \bullet (\mathbf{i} + 2t\mathbf{j})\,dt = 5t^4\,dt,$$

and

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^1 5t^4\,dt = t^5\Big|_0^1 = 1.$$

The third path (c) is made up of two segments, and we parametrize each separately. Let us use y as parameter on the vertical segment (where $x = 0$ and $dx = 0$) and x as parameter on the horizontal segment (where $y = 1$ and $dy = 0$):

$$\begin{aligned}\int_C \mathbf{F} \bullet d\mathbf{r} &= \int_C y^2\,dx + 2xy\,dy \\ &= \int_0^1 (0)\,dy + \int_0^1 (1)\,dx = 1.\end{aligned}$$

In view of these results, we might ask whether $\int_C \mathbf{F} \bullet d\mathbf{r}$ is the same along *every* path from $(0, 0)$ to $(1, 1)$.

Example 2 Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$. Find $\int_C \mathbf{F} \bullet d\mathbf{r}$ from $(1, 0)$ to $(0, -1)$ along

(a) the straight line segment joining these points and

(b) three-quarters of the circle of unit radius centred at the origin and traversed counterclockwise.

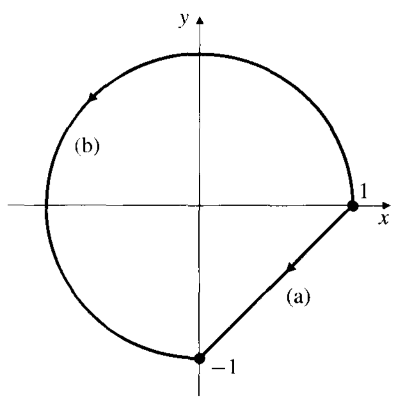

Figure 15.10 Two paths from $(1, 0)$ to $(0, -1)$

Solution Both paths are shown in Figure 15.10. The straight path (a) can be parametrized:

$$\mathbf{r} = (1 - t)\mathbf{i} - t\mathbf{j}, \qquad 0 \le t \le 1.$$

Thus, $d\mathbf{r} = -dt\mathbf{i} - dt\mathbf{j}$, and

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^1 \Big((-t)(-dt) - (1 - t)(-dt)\Big) = \int_0^1 dt = 1.$$

The circular path (b) can be parametrized:

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \qquad 0 \le t \le \frac{3\pi}{2},$$

so that $d\mathbf{r} = -\sin t\,dt\mathbf{i} + \cos t\,dt\mathbf{j}$. Therefore,

$$\mathbf{F} \bullet d\mathbf{r} = -\sin^2 t\,dt - \cos^2 t\,dt = -dt,$$

and we have

$$\int_C \mathbf{F} \bullet d\mathbf{r} = -\int_0^{3\pi/2} dt = -\frac{3\pi}{2}.$$

In this case the line integral depends on the path from $(1, 0)$ to $(0, -1)$ along which the integral is taken.

Some readers may have noticed that in Example 1 above the vector field **F** is conservative, while in Example 2 it is not. Theorem 1 below confirms the link between *independence of path* for a line integral of the tangential component of a vector field and the existence of a scalar potential function for that field. This and subsequent theorems require specific assumptions on the nature of the domain of the vector field **F**, so we need to formulate some topological definitions.

Connected and Simply Connected Domains

Recall that a set S in the plane (or in 3-space) is open if every point in S is the centre of a disk (or a ball) having positive radius and contained in S. If S is open and B is a set (possibly empty) of boundary points of S, then the set $D = S \cup B$ is called a **domain**. A domain cannot contain isolated points. It may be closed, but it must have interior points near any of its boundary points. (See Section 10.1 for a discussion of open and closed sets and interior and boundary points.)

DEFINITION 2

> A domain D is said to be **connected**, if every pair of points P and Q in D can be joined by a piecewise smooth curve lying in D.

For instance, the set of points (x, y) in the plane satisfying $x > 0$, $y > 0$, and $x^2 + y^2 \leq 4$ is a connected domain, but the set of points satisfying $|x| > 1$ is not connected. (There is no path from $(-2, 0)$ to $(2, 0)$ lying entirely in $|x| > 1$.) The set of points (x, y, z) in 3-space satisfying $0 < z < 1$ is a connected domain, but the set satisfying $z \neq 0$ is not.

A closed curve is **simple** if it has no self-intersections other than beginning and ending at the same point. (For example, a circle is a simple closed curve.) Imagine an elastic band stretched in the shape of such a curve. If the elastic is infinitely shrinkable, it can contract down to a single point.

DEFINITION 3

> A **simply connected domain** D is a connected domain in which every *simple closed curve* can be continuously shrunk to a point in D without any part ever passing out of D.

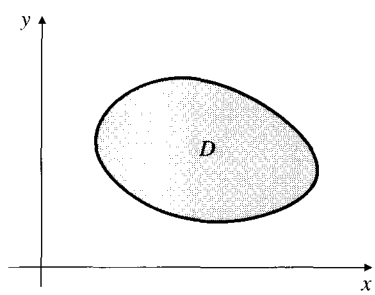

Figure 15.11 A simply connected domain

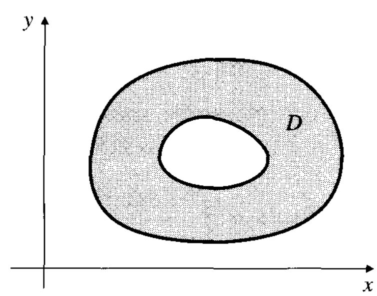

Figure 15.12 A connected domain that is not simply connected

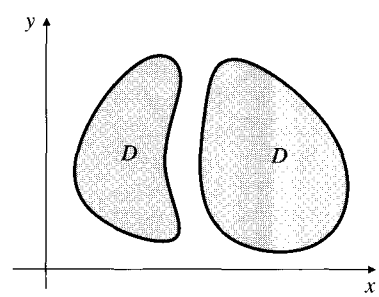

Figure 15.13 A domain that is not connected

Figure 15.11 shows a simply connected domain in the plane. Figure 15.12 shows a connected but not simply connected domain. (A closed curve surrounding the hole cannot be shrunk to a point without passing out of D.) The domain in Figure 15.13 is not even connected. It has two *components*; points in different components cannot be joined by a curve that lies in D.

In the plane, a simply connected domain D can have no holes, not even a hole consisting of a single point. The interior of every non-self-intersecting closed curve in such a domain D lies in D. For instance, the domain of the function $1/(x^2 + y^2)$ is not simply connected because the origin does not belong to it. (The origin is a "hole" in that domain.) In 3-space, a simply connected domain can have holes. The set of all points in $\mathbb{R}^3$ different from the origin is simply connected, as is the exterior of a ball. But the set of all points in $\mathbb{R}^3$ satisfying $x^2 + y^2 > 0$ is not simply connected. Neither is the interior of a doughnut (a *torus*). In general, each of the following conditions characterizes simply connected domains D:

(i) Any simple closed curve in D is the boundary of a "surface" lying in D.

(ii) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves in D having the same endpoints, then $\mathcal{C}_1$ can be continuously deformed into $\mathcal{C}_2$, remaining in D throughout the deformation process.

Independence of Path

THEOREM 1

Independence of path

Let D be an open, connected domain, and let $\mathbf{F}$ be a smooth vector field defined on D. Then the following three statements are *equivalent* in the sense that, if any one of them is true, so are the other two:

(a) $\mathbf{F}$ is conservative in D.

(b) $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = 0$ for every piecewise smooth, closed curve $\mathcal{C}$ in D.

(c) Given any two points P_0 and P_1 in D, $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ has the same value for all piecewise smooth curves in D starting at P_0 and ending at P_1.

PROOF We will show that (a) implies (b), that (b) implies (c), and that (c) implies (a). It then follows that any one implies the other two.

Suppose (a) is true. Then $\mathbf{F} = \nabla\phi$ for some scalar potential function ϕ defined in D. Therefore,

$$\mathbf{F} \bullet d\mathbf{r} = \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right) \bullet \left(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}\right)$$
$$= \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz = d\phi.$$

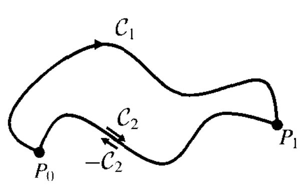

Figure 15.14
$\mathcal{C}_1 - \mathcal{C}_2 = \mathcal{C}_1 + (-\mathcal{C}_2)$ is a closed curve

If $\mathcal{C}$ is any piecewise smooth, closed curve, parametrized, say, by $\mathbf{r} = \mathbf{r}(t)$, $(a \le t \le b)$, then $\mathbf{r}(a) = \mathbf{r}(b)$, and

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_a^b \frac{d\phi(\mathbf{r}(t))}{dt}\,dt = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)) = 0.$$

Thus (a) implies (b).

Now suppose (b) is true. Let P_0 and P_1 be two points in D, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two piecewise smooth curves in D from P_0 to P_1. Let $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ denote the closed curve going from P_0 to P_1 along $\mathcal{C}_1$ and then back to P_0 along $\mathcal{C}_2$ in the opposite direction. (See Figure 15.14.) Since we are assuming that (b) is true, we have

$$0 = \oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \bullet d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F} \bullet d\mathbf{r}.$$

Therefore,

$$\int_{C_1} \mathbf{F} \bullet d\mathbf{r} = \int_{C_2} \mathbf{F} \bullet d\mathbf{r},$$

and we have proved that (b) implies (c).

Finally, suppose that (c) is true. Let $P_0 = (x_0, y_0, z_0)$ be a fixed point in the domain D, and let $P = (x, y, z)$ be an arbitrary point in that domain. Define a function ϕ by

$$\phi(x, y, z) = \int_{C} \mathbf{F} \bullet d\mathbf{r},$$

where $\mathcal{C}$ is some piecewise smooth curve in D from P_0 to P. (Under the hypotheses of the theorem such a curve exists, and, since we are assuming (c), the integral has the same value for all such curves. Therefore, ϕ is well defined in D.) We will show that $\nabla\phi = \mathbf{F}$ and thus establish that $\mathbf{F}$ is conservative and has potential ϕ.

It is sufficient to show that $\partial\phi/\partial x = F_1(x, y, z)$; the other two components are treated similarly. Since D is open, there is a ball of positive radius centred at P and contained in D. Pick a point (x_1, y, z) in this ball having $x_1 < x$. Note that the line from this point to P is parallel to the x-axis. Since we are free to choose the curve $\mathcal{C}$ in the integral defining ϕ, let us choose it to consist of two segments: $\mathcal{C}_1$, which is piecewise smooth and goes from (x_0, y_0, z_0) to (x_1, y, z), and $\mathcal{C}_2$, a straight-line segment from (x_1, y, z) to (x, y, z). (See Figure 15.15.) Then

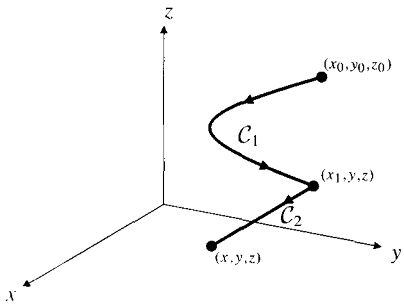

Figure 15.15 A special path from P_0 to P_1

$$\phi(x, y, z) = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r}.$$

The first integral does not depend on x, so its derivative with respect to x is zero. The straight-line path for the second integral is parametrized by $\mathbf{r} = t\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $x_1 \le t \le x$ so $d\mathbf{r} = dt\mathbf{i}$. By the Fundamental Theorem of Calculus,

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \bullet d\mathbf{r} = \frac{\partial}{\partial x} \int_{x_1}^{x} F_1(t, y, z)\, dt = F_1(x, y, z),$$

which is what we wanted. Thus $\mathbf{F} = \nabla\phi$ is conservative, and (c) implies (a).

Remark It is very easy to evaluate the line integral of the tangential component of a *conservative* vector field along a curve $\mathcal{C}$, when you know a potential for $\mathbf{F}$. If $\mathbf{F} = \nabla\phi$, and $\mathcal{C}$ goes from P_0 to P_1, then

$$\int_{C} \mathbf{F} \bullet d\mathbf{r} = \int_{C} d\phi = \phi(P_1) - \phi(P_0).$$

As noted above, the value of the integral depends only on the endpoints of $\mathcal{C}$.

Remark In the next chapter we will add another item to the list of three conditions shown to be equivalent in Theorem 1, provided that the domain D is *simply connected*. For such a domain each of the above three conditions in the theorem is equivalent to

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \text{and} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

We already know that these equations are satisfied on a domain where $\mathbf{F}$ is conservative. Theorem 4 of Section 16.2 states that if these three equations hold on a simply connected domain, then $\mathbf{F}$ is conservative on that domain.

Example 3 For what values of the constants A and B is the vector field

$$\mathbf{F} = Ax\sin(\pi y)\mathbf{i} + \big(x^2\cos(\pi y) + Bye^{-z}\big)\mathbf{j} + y^2e^{-z}\mathbf{k}$$

conservative? For this choice of A and B, evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathcal{C}$ is

(a) the curve $\mathbf{r} = \cos t\mathbf{i} + \sin 2t\mathbf{j} + \sin^2 t\mathbf{k}$, $(0 \le t \le 2\pi)$, and

(b) the curve of intersection of the paraboloid $z = x^2 + 4y^2$ and the plane $z = 3x - 2y$ from $(0, 0, 0)$ to $(1, 1/2, 2)$.

Solution $\mathbf{F}$ cannot be conservative unless

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \text{and} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y},$$

that is, unless

$$A\pi x\cos(\pi y) = 2x\cos(\pi y), \qquad 0 = 0, \qquad \text{and} \qquad -Bye^{-z} = 2ye^{-z}.$$

Thus, we require that $A = 2/\pi$ and $B = -2$. In this case, it is easily checked that

$$\mathbf{F} = \nabla\phi, \quad \text{where} \quad \phi = \left(\frac{x^2\sin(\pi y)}{\pi} - y^2e^{-z}\right).$$

For the curve (a) we have $\mathbf{r}(0) = \mathbf{i} = \mathbf{r}(2\pi)$, so this curve is a closed curve, and

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \nabla\phi \bullet d\mathbf{r} = 0.$$

Since the curve (b) starts at $(0, 0, 0)$ and ends at $(1, 1/2, 2)$, we have

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \left(\frac{x^2\sin(\pi y)}{\pi} - y^2e^{-z}\right)\Bigg|_{(0,0,0)}^{(1,1/2,2)} = \frac{1}{\pi} - \frac{1}{4e^2}.$$

The following example shows how to exploit the fact that

$$\int_C \mathbf{F} \bullet d\mathbf{r}$$

is easily evaluated for conservative $\mathbf{F}$ even if the $\mathbf{F}$ we want to integrate isn't quite conservative.

Example 4 Evaluate $I = \oint_C (e^x\sin y + 3y)dx + (e^x\cos y + 2x - 2y)dy$ counterclockwise around the ellipse $4x^2 + y^2 = 4$.

Solution $I = \oint_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F}$ is the vector field

$$\mathbf{F} = \big(e^x\sin y + 3y\big)\mathbf{i} + \big(e^x\cos y + 2x - 2y\big)\mathbf{j}.$$

This vector field is not conservative, but it would be if the $3y$ term in F_1 were $2y$ instead; specifically, if

$$\phi(x, y) = e^x \sin y + 2xy - y^2,$$

then $\mathbf{F} = \nabla\phi + y\mathbf{i}$, the sum of a conservative part and a nonconservative part. Therefore, we have

$$I = \oint_C \nabla\phi \bullet d\mathbf{r} + \oint_C y\,dx.$$

The first integral is zero since $\nabla\phi$ is conservative and $\mathcal{C}$ is closed. For the second integral we parametrize $\mathcal{C}$ by $x = \cos t$, $y = 2\sin t$, $(0 \le t \le 2\pi)$, and obtain

$$I = \oint_C y\,dx = -2\int_0^{2\pi} \sin^2 t\,dt = -2\int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt = -2\pi.$$

Exercises 15.4

In Exercises 1–6, evaluate the line integral of the tangential component of the given vector field along the given curve.

1. $\mathbf{F}(x, y) = xy\mathbf{i} - x^2\mathbf{j}$ along $y = x^2$ from $(0, 0)$ to $(1, 1)$
2. $\mathbf{F}(x, y) = \cos x\,\mathbf{i} - y\mathbf{j}$ along $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$
3. $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$ along the straight line from $(0, 0, 0)$ to $(1, 1, 1)$
4. $\mathbf{F}(x, y, z) = z\mathbf{i} - y\mathbf{j} + 2x\mathbf{k}$ along the curve $x = t$, $y = t^2$, $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$
5. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ from $(-1, 0, 0)$ to $(1, 0, 0)$ along either direction of the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y$
6. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} - (x + y)\mathbf{k}$ along the polygonal path from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$
7. Find the work done by the force field
$$\mathbf{F} = (x + y)\mathbf{i} + (x - z)\mathbf{j} + (z - y)\mathbf{k}$$
in moving an object from $(1, 0, -1)$ to $(0, -2, 3)$ along any smooth curve.
8. Evaluate $\oint_C x^2y^2\,dx + x^3y\,dy$ counterclockwise around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.
9. Evaluate
$$\int_C e^{x+y}\sin(y + z)\,dx + e^{x+y}\big(\sin(y + z) + \cos(y + z)\big)\,dy + e^{x+y}\cos(y + z)\,dz$$
along the straight line segment from (0,0,0) to $(1, \frac{\pi}{4}, \frac{\pi}{4})$.
10. The field $\mathbf{F} = (axy + z)\mathbf{i} + x^2\mathbf{j} + (bx + 2z)\mathbf{k}$ is conservative. Find a and b, and find a potential for $\mathbf{F}$. Also, evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathcal{C}$ is the curve from $(1, 1, 0)$ to $(0, 0, 3)$ that lies on the intersection of the surfaces $2x + y + z = 3$ and $9x^2 + 9y^2 + 2z^2 = 18$ in the octant $x \ge 0$, $y \ge 0$, $z \ge 0$.
11. Determine the values of A and B for which the vector field
$$\mathbf{F} = Ax\ln z\,\mathbf{i} + By^2z\,\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k}$$
is conservative. If $\mathcal{C}$ is the straight line from $(1, 1, 1)$ to $(2, 1, 2)$, find
$$\int_C 2x\ln z\,dx + 2y^2z\,dy + y^3\,dz.$$
12. Find the work done by the force field
$$\mathbf{F} = (y^2\cos x + z^3)\mathbf{i} + (2y\sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k}$$
in moving a particle along the curve $x = \sin^{-1}t$, $y = 1 - 2t$, $z = 3t - 1$, $(0 \le t \le 1)$.
13. If $\mathcal{C}$ is the intersection of $z = \ln(1 + x)$ and $y = x$ from $(0, 0, 0)$ to $(1, 1, \ln 2)$, evaluate
$$\int_C \big(2x\sin(\pi y) - e^z\big)\,dx + \big(\pi x^2\cos(\pi y) - 3e^z\big)\,dy - xe^z\,dz.$$
14. Is each of the following sets a domain? a connected domain? a simply connected domain?

(a) the set of points (x, y) in the plane such that $x > 0$ and $y \geq 0$

(b) the set of points (x, y) in the plane such that $x = 0$ and $y \geq 0$

(c) the set of points (x, y) in the plane such that $x \neq 0$ and $y > 0$

(d) the set of points (x, y, z) in 3-space such that $x^2 > 1$

(e) the set of points (x, y, z) in 3-space such that $x^2 + y^2 > 1$

(f) the set of points (x, y, z) in 3-space such that $x^2 + y^2 + z^2 > 1$

In Exercises 15–19, evaluate the closed line integrals

$$\text{(a)} \quad \oint_C x\,dy, \qquad \text{(b)} \quad \oint_C y\,dx$$

around the given curves, all oriented counterclockwise.

15. The circle $x^2 + y^2 = a^2$

16. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

17. The boundary of the half-disk $x^2 + y^2 \leq a^2$, $y \geq 0$

18. The boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$

19. The triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$

20. On the basis of your results for Exercises 15–19, guess the values of the closed line integrals

$$\text{(a)} \quad \oint_C x\,dy, \qquad \text{(b)} \quad \oint_C y\,dx$$

for any non-self-intersecting closed curve in the xy-plane. Prove your guess in the case that $\mathcal{C}$ bounds a region of the plane that is both x-simple and y-simple. (See Section 14.2.)

21. If f and g are scalar fields with continuous first partial derivatives in a connected domain D, show that

$$\int_C f\nabla g \bullet d\mathbf{r} + \int_C g\nabla f \bullet d\mathbf{r} = f(Q)g(Q) - f(P)g(P)$$

for any piecewise smooth curve in D from P to Q.

22. Evaluate

$$\frac{1}{2\pi}\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

(a) counterclockwise around the circle $x^2 + y^2 = a^2$,

(b) clockwise around the square with vertices $(-1, -1)$, $(-1, 1)$, $(1, 1)$, and $(1, -1)$,

(c) counterclockwise around the boundary of the region $1 \leq x^2 + y^2 \leq 4$, $y \geq 0$.

23. Review Example 5 in Section 15.2 in which it was shown that

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right),$$

for all $(x, y) \neq (0, 0)$. Why does this result, together with that of Exercise 22, not contradict the final assertion in the remark following Theorem 1?

* **24.** Let $\mathcal{C}$ be a piecewise smooth curve in the xy-plane which does not pass through the origin. Let $\theta = \theta(x, y)$ be the polar angle coordinate of the point $P = (x, y)$ on $\mathcal{C}$, not restricted to an interval of length 2π, but varying continuously as P moves from one end of $\mathcal{C}$ to the other. As in Example 5 of Section 15.2, it happens that

$$\nabla\theta = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

If, in addition, $\mathcal{C}$ is a closed curve, show that

$$w(\mathcal{C}) = \frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2 + y^2}$$

has an integer value. w is called the **winding number** of $\mathcal{C}$ about the origin.

15.5 Surfaces and Surface Integrals

This section and the next are devoted to integrals of functions defined over surfaces in 3-space. Before we can begin, it is necessary to make more precise just what is meant by the term "surface." Until now we have been treating surfaces in an intuitive way, either as the graphs of functions $f(x, y)$ or as the graphs of equations $f(x, y, z) = 0$.

A smooth curve is a *one-dimensional* object because points on it can be located by giving *one coordinate* (for instance, the distance from an endpoint). Therefore, the curve can be defined as the range of a vector-valued function of one real variable. A surface is a *two-dimensional* object; points on it can be located by using *two coordinates*, and it can be defined as the range of a vector-valued function of two real variables. We will call certain such functions parametric surfaces.

Parametric Surfaces

DEFINITION 4

A **parametric surface** in 3-space is a continuous function $\mathbf{r}$ defined on some rectangle R given by $a \le u \le b$, $c \le v \le d$ in the uv-plane and having values in 3-space:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \qquad (u, v) \text{ in } R.$$

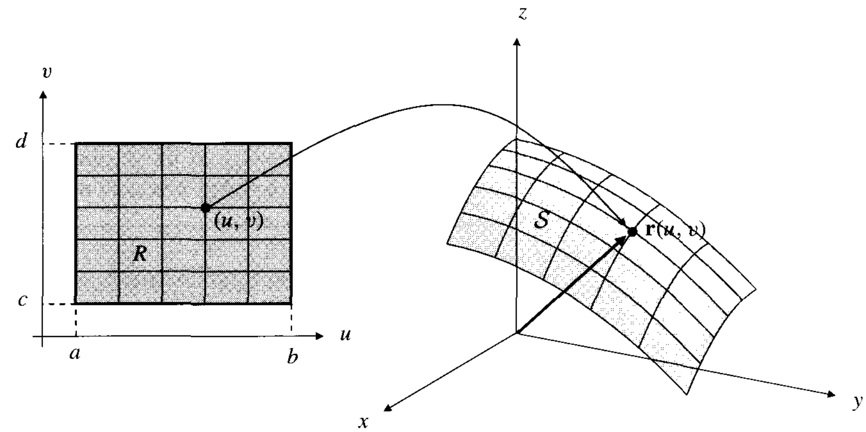

Figure 15.16 A parametric surface $\mathcal{S}$ defined on parameter region R. The *contour curves* on $\mathcal{S}$ correspond to the rulings of R

Actually, we think of the *range* of the function $\mathbf{r}(u, v)$ as being the parametric surface. It is a set $\mathcal{S}$ of points (x, y, z) in 3-space whose position vectors are the vectors $\mathbf{r}(u, v)$ for (u, v) in R. (See Figure 15.16.) If $\mathbf{r}$ is one-to-one, then the surface does not intersect itself. In this case $\mathbf{r}$ maps the boundary of the rectangle R (the four edges) onto a curve in 3-space, which we call the **boundary of the parametric surface**. The requirement that R be a rectangle is made only to simplify the discussion. Any connected, closed, bounded set in the uv-plane, having well-defined area and consisting of an open set together with its boundary points would do as well. Thus, we will from time to time consider parametric surfaces over closed disks, triangles, or other such domains in the uv-plane. Being the range of a continuous function defined on a closed, bounded set, a parametric surface is always bounded in 3-space.

Example 1 The graph of $z = f(x, y)$, where f has the rectangle R as its domain, can be represented as the parametric surface

$$\mathbf{r} = \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}$$

for (u, v) in R. Its scalar parametric equations are

$$x = u, \qquad y = v, \qquad z = f(u, v), \qquad (u, v) \text{ in } R.$$

For such graphs it is sometimes convenient to identify the uv-plane with the xy-plane and write the equation of the surface in the form

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}, \qquad (x, y) \text{ in } R.$$

Example 2 Describe the surface

$$\mathbf{r} = a\cos u \sin v\,\mathbf{i} + a\sin u \sin v\,\mathbf{j} + a\cos v\,\mathbf{k}, \qquad (0 \le u \le 2\pi,\ 0 \le v \le \pi/2),$$

where $a > 0$. What is its boundary?

Solution Observe that if $x = a\cos u \sin v$, $y = a\sin u \sin v$, and $z = a\cos v$, then $x^2 + y^2 + z^2 = a^2$. Thus, the given parametric surface lies on the sphere of radius a centred at the origin. (Observe that u and v are the spherical coordinates θ and ϕ on the sphere.) The restrictions on u and v allow (x, y) to be any point in the disk $x^2 + y^2 \le a^2$ but force $z \ge 0$. Thus, the surface is the *upper half* of the sphere. The given parametrization is one-to-one on the open rectangle $0 < u < 2\pi$, $0 < v < \pi/2$, but not on the closed rectangle, since the edges $u = 0$ and $u = 2\pi$ get mapped onto the same points, and the entire edge $v = 0$ collapses to a single point. The boundary of the surface is still the circle $x^2 + y^2 = a^2$, $z = 0$, and corresponds to the edge $v = \pi/2$ of the rectangle.

Remark Surface parametrizations that are one-to-one only in the interior of the parameter domain R are still reasonable representations of the surface. However, as in Example 2, the boundary of the surface may be obtained from only part of the boundary of R, or there may be no boundary at all, in which case the surface is called a **closed surface**. For example, if the domain of $\mathbf{r}$ in Example 2 is extended to allow $0 \le v \le \pi$, then the surface becomes the entire sphere of radius a centred at the origin. The sphere is a closed surface, having no boundary curves.

Remark Like parametrizations of curves, parametrizations of surfaces are not unique. The hemisphere in Example 2 can also be parametrized:

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{a^2 - u^2 - v^2}\,\mathbf{k} \qquad \text{for} \quad u^2 + v^2 \le a^2.$$

Here, the domain of $\mathbf{r}$ is a closed disk of radius a.

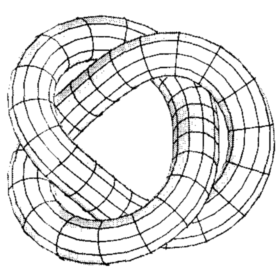

Figure 15.17 A tube in the shape of a trefoil knot

Example 3 **(A tube around a curve)** If $\mathbf{r} = \mathbf{F}(t)$, $a \le t \le b$, is a parametric curve $\mathcal{C}$ in 3-space having unit normal $\hat{\mathbf{N}}(t)$ and binormal $\hat{\mathbf{B}}(t)$, then the parametric surface

$$\mathbf{r} = \mathbf{F}(u) + s\cos v\,\hat{\mathbf{N}}(u) + s\sin v\,\hat{\mathbf{B}}(u), \quad a \le u \le b, \quad 0 \le v \le 2\pi,$$

is a tube-shaped surface of radius s centred along the curve $\mathcal{C}$. (Why?) Figure 15.17 shows such a tube, having radius $s = 0.25$, around the curve

$$\mathbf{r} = \big(1 + 0.3\cos(3t)\big)\big(\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}\big) + 0.35\sin(3t)\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

This closed curve is called a **trefoil knot**.

Composite Surfaces

If two parametric surfaces are joined together along part or all of their boundary curves, the result is called a **composite surface**, or, thinking geometrically, just a **surface**. For example, a sphere can be obtained by joining two hemispheres along their boundary circles. In general, composite surfaces can be obtained by joining a finite number of parametric surfaces pairwise along edges. The surface of a cube consists of the six square faces joined in pairs along the edges of the cube. This surface is closed since there are no unjoined edges to comprise the boundary. If the top square face is removed, the remaining five form the surface of a cubical box with no top. The top edges of the four side faces now constitute the boundary of this composite surface. (See Figure 15.18.)

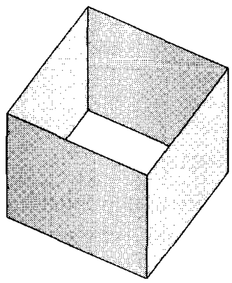

Figure 15.18 A composite surface obtained by joining five smooth parametric surfaces (squares) in pairs along edges. The four unpaired edges at the tops of the side faces make up the boundary of the composite surface

Surface Integrals

In order to define integrals of functions defined on a surface as limits of Riemann sums, we need to refer to the *areas* of regions on the surface. It is more difficult to define the area of a curved surface than it is to define the length of a curve. However, you will likely have a good idea of what area means for a region lying in a plane, and we examined briefly the problem of finding the area of the graph of a function $f(x, y)$ in Section 14.7. We will avoid difficulties by assuming that all the surfaces we will encounter are "smooth enough" that they can be subdivided into small pieces each of which is approximately planar. We can then approximate the surface area of each piece by a plane area and add up the approximations to get a Riemann sum approximation to the area of the whole surface. We will make more precise definitions of "smooth surface" and "surface area" later in this section. For the moment, we assume the reader has an intuitive feel for what they mean.

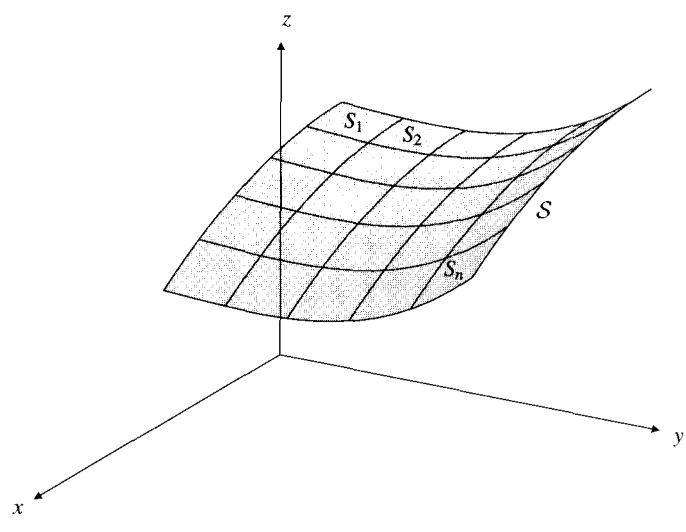

Figure 15.19 A partition of a parametric surface into many nonoverlapping pieces

Let $\mathcal{S}$ be a smooth surface of finite area in $\mathbb{R}^3$, and let $f(x, y, z)$ be a bounded function defined at all points of $\mathcal{S}$. If we subdivide $\mathcal{S}$ into small, nonoverlapping pieces, say $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$, where $\mathcal{S}_i$ has area ΔS_i (see Figure 15.19), we can form a **Riemann sum** R_n for f on $\mathcal{S}$ by choosing arbitrary points (x_i, y_i, z_i) in $\mathcal{S}_i$ and letting

$$R_n = \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta S_i.$$

If such Riemann sums have a unique limit as the diameters of all the pieces S_i approach zero, independently of how the points (x_i, y_i, z_i) are chosen, then we say that f is **integrable** on $\mathcal{S}$ and denote the limit by

$$\iint_{\mathcal{S}} f(x, y, z)\, dS.$$

Smooth Surfaces, Normals, and Area Elements

A surface is smooth if it has a unique tangent plane at any nonboundary point P. A nonzero vector $\mathbf{n}$ normal to that tangent plane at P is said to be normal to the surface at P. The following somewhat technical definition makes this precise.

DEFINITION 5

A set $\mathcal{S}$ in 3-space is a **smooth surface** if any point P in $\mathcal{S}$ has a neighbourhood N (an open ball of positive radius centred at P) that is the domain of a smooth function $g(x, y, z)$ satisfying:

(i) $N \cap S = \{Q \in N : g(Q) = 0\}$ and

(ii) $\nabla g(Q) \neq \mathbf{0}$, if Q is in $N \cap S$.

For example, the cone $x^2 + y^2 = z^2$, with the origin removed, is a smooth surface. Note that $\nabla(x^2 + y^2 - z^2) = \mathbf{0}$ at the origin, and the cone is not smooth there, since it does not have a unique tangent plane.

A parametric surface cannot satisfy the condition of the smoothness definition at its boundary points but will be called **smooth** if that condition is satisfied at all nonboundary points.

We can find the normal to a smooth parametric surface defined on parameter domain R as follows. If (u_0, v_0) is a point in the interior of R, then $\mathbf{r} = \mathbf{r}(u, v_0)$ and $\mathbf{r} = \mathbf{r}(u_0, v)$ are two curves on $\mathcal{S}$, intersecting at $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$ and having, at that point, tangent vectors

$$\left.\frac{\partial \mathbf{r}}{\partial u}\right|_{(u_0, v_0)} \quad \text{and} \quad \left.\frac{\partial \mathbf{r}}{\partial v}\right|_{(u_0, v_0)},$$

respectively. Assuming these two tangent vectors are not parallel, their cross product $\mathbf{n}$, which is not zero, is *normal* to $\mathcal{S}$ at $\mathbf{r}_0$. Furthermore, the *area element* on $\mathcal{S}$ bounded by the four curves $\mathbf{r} = \mathbf{r}(u_0, v)$, $\mathbf{r} = \mathbf{r}(u_0 + du, v)$, $\mathbf{r} = \mathbf{r}(u, v_0)$, and $\mathbf{r} = \mathbf{r}(u, v_0 + dv)$ (see Figure 15.20) is an infinitesimal parallelogram spanned by the vectors $(\partial \mathbf{r}/\partial u)\, du$ and $(\partial \mathbf{r}/\partial v)\, dv$ (at (u_0, v_0)), and hence has area

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv.$$

Let us express the normal vector $\mathbf{n}$ and the area element dS in terms of the components of $\mathbf{r}$. Since

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k},$$

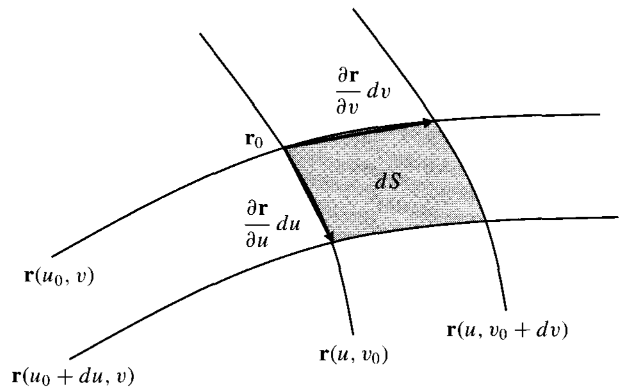

Figure 15.20 An area element dS on a parametric surface

the **normal vector** to $\mathcal{S}$ at $\mathbf{r}(u, v)$ is

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \frac{\partial(y, z)}{\partial(u, v)}\mathbf{i} + \frac{\partial(z, x)}{\partial(u, v)}\mathbf{j} + \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k}.$$

Also, the **area element** at a point $\mathbf{r}(u, v)$ on the surface is given by

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv$$

$$= \sqrt{\left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2}\, du\, dv.$$

The area of the surface itself is the "sum" of these area elements:

$$\text{Area of } \mathcal{S} = \iint_{\mathcal{S}} dS.$$

Example 4 The graph $z = g(x, y)$ of a function g with continuous first partial derivatives in a domain D of the xy-plane can be regarded as a parametric surface $\mathcal{S}$ with parametrization

$$x = u, \qquad y = v, \qquad z = g(u, v), \qquad (u, v) \text{ in } D.$$

In this case

$$\frac{\partial(y, z)}{\partial(u, v)} = -g_1(u, v), \qquad \frac{\partial(z, x)}{\partial(u, v)} = -g_2(u, v), \quad \text{and} \quad \frac{\partial(x, y)}{\partial(u, v)} = 1,$$

and, since the parameter region coincides with the domain D of g, the surface integral of $f(x, y, z)$ over $\mathcal{S}$ can be expressed as a double integral over D:

$$\iint_S f(x, y, z)\, dS$$
$$= \iint_D f\big(x, y, g(x, y)\big)\sqrt{1 + \big(g_1(x, y)\big)^2 + \big(g_2(x, y)\big)^2}\, dx\, dy.$$

As observed in Section 14.7, this formula can also be justified geometrically. The vector $\mathbf{n} = -g_1(x, y)\mathbf{i} - g_2(x, y)\mathbf{j} + \mathbf{k}$ is normal to $\mathcal{S}$ and makes angle γ with the positive z-axis, where

$$\cos\gamma = \frac{\mathbf{n} \bullet \mathbf{k}}{|\mathbf{n}|} = \frac{1}{\sqrt{1 + \big(g_1(x, y)\big)^2 + \big(g_2(x, y)\big)^2}}.$$

The surface area element dS must have area $1/\cos\gamma$ times the area $dx\, dy$ of its perpendicular projection onto the xy-plane. (See Figure 15.21.) ■

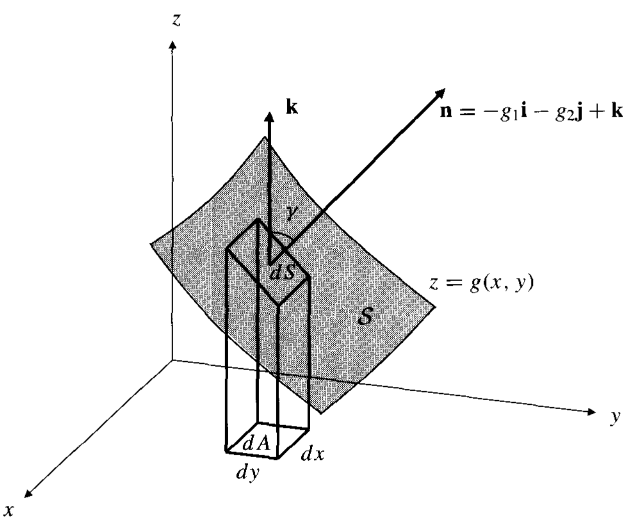

Figure 15.21 The surface area element dS and its projection onto the xy-plane

Evaluating Surface Integrals

We illustrate the use of the formulas given above for dS in calculating surface integrals.

Example 5 Evaluate $\iint_S z\, dS$ over the conical surface $z = \sqrt{x^2 + y^2}$ between $z = 0$ and $z = 1$.

Solution Since $z^2 = x^2 + y^2$ on the surface $\mathcal{S}$, we have $\partial z/\partial x = x/z$ and $\partial z/\partial y = y/z$. Therefore,

$$dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dx\, dy = \sqrt{\frac{z^2 + z^2}{z^2}}\, dx\, dy = \sqrt{2}\, dx\, dy.$$

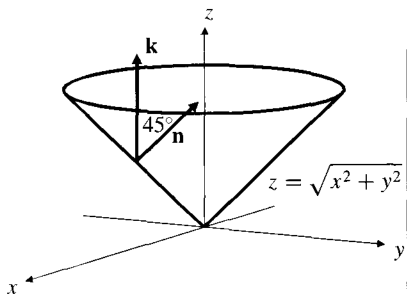

Figure 15.22 $dS = \sqrt{2}\,dx\,dy$ on this cone

(Note that we could have anticipated this result, since the normal to the cone always makes an angle of $\gamma = 45°$ with the positive z-axis; see Figure 15.22. Therefore, $dS = dx\,dy/\cos 45° = \sqrt{2}\,dx\,dy$.) Since $z = \sqrt{x^2+y^2} = r$ on the conical surface, it is easiest to carry out the integration in polar coordinates:

$$\iint_S z\,dS = \sqrt{2}\iint_{x^2+y^2\le 1} z\,dx\,dy$$
$$= \sqrt{2}\int_0^{2\pi} d\theta \int_0^1 r^2\,dr = \frac{2\sqrt{2}\pi}{3}.$$

Example 6 Find the moment of inertia about the z-axis of the parametric surface $x = 2uv$, $y = u^2 - v^2$, $z = u^2 + v^2$, where $u^2 + v^2 \le 1$.

Solution We are asked to find $\iint_S (x^2+y^2)\,dS$. We have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2v & 2u \\ 2u & -2v \end{vmatrix} = -4(u^2+v^2),$$
$$\frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} 2u & 2v \\ 2v & 2u \end{vmatrix} = 4(u^2-v^2),$$
$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 8uv.$$

Therefore, the surface area element on $\mathcal{S}$ is given by

$$dS = 4\sqrt{(u^2+v^2)^2 + (u^2-v^2)^2 + 4u^2v^2}\,du\,dv$$
$$= 4\sqrt{2(u^4+v^4+2u^2v^2)}\,du\,dv = 4\sqrt{2}(u^2+v^2)\,du\,dv.$$

Now $x^2+y^2 = 4u^2v^2 + (u^2-v^2)^2 = (u^2+v^2)^2$. Thus,

$$\iint_S (x^2+y^2)\,dS = \iint_{u^2+v^2\le 1} (u^2+v^2)^2\, 4\sqrt{2}(u^2+v^2)\,du\,dv$$
$$= 4\sqrt{2}\int_0^{2\pi} d\theta \int_0^1 r^6\, r\,dr \qquad \text{(using polar coordinates).}$$
$$= \sqrt{2}\pi.$$

This is the required moment of inertia.

Even though most surfaces we encounter can be easily parametrized, it is usually possible to obtain the surface area element dS geometrically rather than relying on the parametric formula. As we have seen above, if a surface has a one-to-one projection onto a region in the xy-plane, then the area element dS on the surface can be expressed as

$$dS = \left|\frac{1}{\cos\gamma}\right| dx\,dy = \frac{|\mathbf{n}|}{|\mathbf{n}\bullet\mathbf{k}|}\,dx\,dy,$$

where γ is the angle between the normal vector $\mathbf{n}$ to $\mathcal{S}$ and the positive z-axis. This formula is useful no matter how we obtain $\mathbf{n}$.

Consider a surface $\mathcal{S}$ with equation of the form $G(x, y, z) = 0$. As we discovered in Section 12.7, if G has continuous first partial derivatives that do not all vanish at a point (x, y, z) on $\mathcal{S}$, then the nonzero vector

$$\mathbf{n} = \nabla G(x, y, z)$$

is normal to $\mathcal{S}$ at that point. Since $\mathbf{n} \bullet \mathbf{k} = G_3(x, y, z)$, if $\mathcal{S}$ has a one-to-one projection onto the domain D in the xy-plane, then

$$dS = \left| \frac{\nabla G(x, y, z)}{G_3(x, y, z)} \right| dx\, dy,$$

and the surface integral of $f(x, y, z)$ over $\mathcal{S}$ can be expressed as a double integral over the domain D:

$$\iint_{\mathcal{S}} f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \left| \frac{\nabla G(x, y, z)}{G_3(x, y, z)} \right| dx\, dy.$$

Of course, there are analogous formulas for area elements of surfaces (and integrals over surfaces) with one-to-one projections onto the xz-plane or the yz-plane. (G_3 is replaced by G_2 and G_1, respectively.)

Example 7 Find the moment about $z = 0$, that is, $\iint_{\mathcal{S}} z\, dS$, where $\mathcal{S}$ is the hyperbolic bowl $z^2 = 1 + x^2 + y^2$ between the planes $z = 1$ and $z = \sqrt{5}$.

Solution $\mathcal{S}$ is given by $G(x, y, z) = 0$, where $G(x, y, z) = x^2 + y^2 - z^2 + 1$. It lies above the disk $x^2 + y^2 \le 4$ in the xy-plane. We have $\nabla G = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$, and $G_3 = -2z$. Hence, on $\mathcal{S}$, we have

$$z\, dS = z \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z}\, dx\, dy = \sqrt{1 + 2(x^2 + y^2)}\, dx\, dy,$$

and the required moment is

$$\begin{aligned} \iint_{\mathcal{S}} z\, dS &= \iint_{x^2+y^2 \le 4} \sqrt{1 + 2(x^2 + y^2)}\, dx\, dy \\ &= \int_0^{2\pi} d\theta \int_0^2 \sqrt{1 + 2r^2}\, r\, dr = \frac{\pi}{3}(1 + 2r^2)^{3/2}\Big|_0^2 = \frac{26\pi}{3}. \end{aligned}$$

■

The next example illustrates a technique that can often reduce the effort needed to integrate over a cylindrical surface.

Example 8 Find the area of that part of the cylinder $x^2 + y^2 = 2ay$ that lies inside the sphere $x^2 + y^2 + z^2 = 4a^2$.

Solution One quarter of the required area lies in the first octant. (See Figure 15.23.) Since the cylinder is generated by vertical lines, we can express an area element dS on it in terms of the length element ds along the curve C in the xy-plane having equation $x^2 + y^2 = 2ay$:

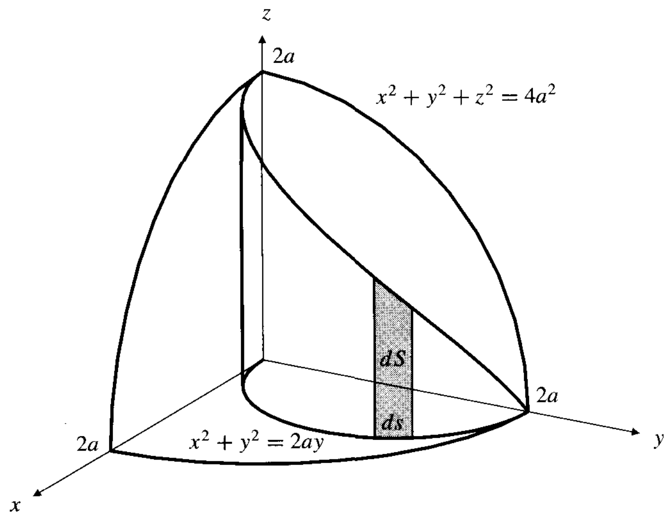

Figure 15.23 An area element on a cylinder. The z-coordinate has already been integrated

$$dS = z\,ds = \sqrt{4a^2 - x^2 - y^2}\,ds.$$

In expressing dS this way, we have already integrated dz, so only a single integral is needed to sum these area elements. Again, it is convenient to use polar coordinates in the xy-plane. In terms of polar coordinates, the curve C has equation $r = 2a\sin\theta$. Thus $dr/d\theta = 2a\cos\theta$ and $ds = \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = 2a\,d\theta$. Therefore, the total surface area of that part of the cylinder that lies inside the sphere is given by

$$\begin{aligned} A &= 4\int_0^{\pi/2} \sqrt{4a^2 - r^2}\; 2a\,d\theta \\ &= 8a\int_0^{\pi/2} \sqrt{4a^2 - 4a^2\sin^2\theta}\,d\theta \\ &= 16a^2\int_0^{\pi/2} \cos\theta\,d\theta = 16a^2 \text{ square units.} \end{aligned}$$

Remark The area calculated in Example 8 can also be calculated by projecting the cylindrical surface in Figure 15.23 into the yz-plane. (This is the only coordinate plane you can use. Why?) See Exercise 6 below.

In spherical coordinates, ϕ and θ can be used as parameters on the spherical surface $\rho = a$. The area element on that surface can therefore be expressed in terms of these coordinates:

Area element on the sphere $\rho = a$: $\quad dS = a^2 \sin\phi\,d\phi\,d\theta.$

(See Figure 14.43 in Section 14.6 and Exercise 2 below.)

Example 9 Find $\displaystyle\iint_S z^2\,dS$ over the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.

Solution Since $z = a\cos\phi$ and the hemisphere corresponds to $0 \le \theta \le 2\pi$, and $0 \le \phi \le \dfrac{\pi}{2}$, we have

$$\iint_S z^2\,dS = \int_0^{2\pi} d\theta \int_0^{\pi/2} a^2\cos^2\phi\, a^2\sin\phi\,d\phi$$
$$= 2\pi a^4\left(-\frac{1}{3}\cos^3\phi\right)\Big|_0^{\pi/2} = \frac{2\pi a^4}{3}.$$

Finally, if a composite surface $\mathcal{S}$ is composed of *smooth parametric surfaces* joined pairwise along their edges, then we call $\mathcal{S}$ a **piecewise smooth surface**. The surface integral of a function f over a piecewise smooth surface $\mathcal{S}$ is the sum of the surface integrals of f over the individual smooth surfaces comprising $\mathcal{S}$. We will encounter an example of this in the next section.

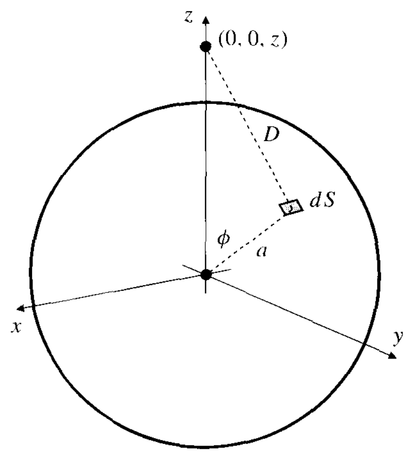

Figure 15.24 The attraction of a sphere

The Attraction of a Spherical Shell

In Section 14.7 we calculated the gravitational attraction of a disk in the xy-plane on a mass m located at position $(0, 0, b)$ on the z-axis. Here, we undertake a similar calculation of the attractive force exerted on m by a spherical shell of radius a and areal density σ (units of mass per unit area) centred at the origin. This calculation would be more difficult if we tried to do it by integrating the vertical component of the force on m as we did in Section 14.7. It is greatly simplified if, instead, we use an integral to find the total *gravitational potential* $\Phi(0, 0, z)$ due to the sphere at position $(0, 0, z)$ and then calculate the force on m as $\mathbf{F} = m\nabla\Phi(0, 0, b)$.

By the Cosine Law, the distance from the point with spherical coordinates $[a, \phi, \theta]$ to the point $(0, 0, z)$ on the positive z-axis (see Figure 15.24) is

$$D = \sqrt{a^2 + z^2 - 2az\cos\phi}.$$

The area element $dS = a^2\sin\phi\,d\phi\,d\theta$ at $[a, \phi, \theta]$ has mass $dm = \sigma\,dS$, and its gravitational potential at $(0, 0, z)$ (see Example 1 in Section 15.2) is

$$d\Phi(0, 0, z) = \frac{k\,dm}{D} = \frac{k\sigma a^2\sin\phi\,d\phi\,d\theta}{\sqrt{a^2 + z^2 - 2az\cos\phi}}.$$

For the total potential at $(0, 0, z)$ due to the sphere, we integrate $d\Phi$ over the surface of the sphere. Making the change of variables $u = a^2 + z^2 - 2az\cos\phi$, $du = 2az\sin\phi\,d\phi$, we obtain

$$\Phi(0, 0, z) = k\sigma a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \frac{\sin\phi\,d\phi}{\sqrt{a^2 + z^2 - 2az\cos\phi}}$$
$$= 2\pi k\sigma a^2 \int_{(z-a)^2}^{(z+a)^2} \frac{1}{\sqrt{u}}\,\frac{du}{2az}$$
$$= \frac{2\pi k\sigma a}{z}\sqrt{u}\,\Big|_{(z-a)^2}^{(z+a)^2}$$
$$= \frac{2\pi k\sigma a}{z}\Big(z + a - |z - a|\Big) = \begin{cases} 4\pi k\sigma a^2/z & \text{if } z > a \\ 4\pi k\sigma a & \text{if } z < a. \end{cases}$$

The potential is constant inside the sphere and decreases proportionally to $1/z$ outside. The force on a mass m located at $(0, 0, b)$ is, therefore,

$$\mathbf{F} = m\nabla\Phi(0, 0, b) = \begin{cases} -(4\pi k m\sigma a^2/b^2)\mathbf{k} & \text{if } b > a \\ \mathbf{0} & \text{if } b < a. \end{cases}$$

We are led to the somewhat surprising result that, if the mass m is anywhere inside the sphere, the net force of attraction of the sphere on it is zero. This is to be expected at the centre of the sphere, but away from the centre it appears that the larger forces due to parts of the sphere close to m are exactly cancelled by smaller forces due to parts farther away; these farther parts have larger area and therefore larger total mass. If m is outside the sphere, the sphere attracts it with a force of magnitude

$$F = \frac{kmM}{b^2},$$

where $M = 4\pi\sigma a^2$ is the total mass of the sphere. This is the same force that would be exerted by a point mass with the same mass as the sphere and located at the centre of the sphere.

Remark A solid ball of constant density, or density depending only on the distance from the centre (for instance, a planet), can be regarded as being made up of mass elements that are concentric spheres of constant density. Therefore, the attraction of such a ball on a mass m located outside the ball will also be the same as if the whole mass of the ball were concentrated at its centre. However, the attraction on a mass m located somewhere inside the ball will be that produced by only the part of the ball that is closer to the centre than m is. The maximum force of attraction will occur when m is right at the surface of the ball. If the density is constant, the magnitude of the force increases linearly with the distance from the centre (why?) up to the surface and then decreases with the square of the distance as m recedes from the ball. (See Figure 15.25.)

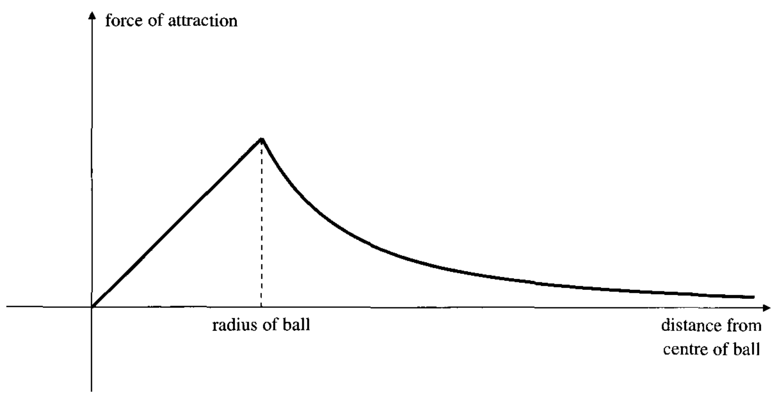

Figure 15.25 The force of attraction of a homogeneous solid ball on a particle located at varying distances from the centre of the ball

Remark All of the above discussion also holds for the electrostatic attraction or repulsion of a point charge by a uniform charge density over a spherical shell, which is also governed by an inverse square law. In particular, there is no net electrostatic force on a charge located inside the shell.

Exercises 15.5

1. Verify that on the curve with polar equation $r = g(\theta)$ the arc length element is given by

$$ds = \sqrt{(g(\theta))^2 + (g'(\theta))^2}\, d\theta.$$

What is the area element on the vertical cylinder given in terms of cylindrical coordinates by $r = g(\theta)$?

2. Verify that on the spherical surface $x^2 + y^2 + z^2 = a^2$ the area element is given in terms of spherical coordinates by

$dS = a^2 \sin\phi \, d\phi \, d\theta$.

3. Find the area of the part of the plane $Ax + By + Cz = D$ lying inside the elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ay$.

5. State formulas for the surface area element dS for the surface with equation $F(x, y, z) = 0$ valid for the case where the surface has a one-to-one projection on (a) the xz-plane and (b) the yz-plane.

6. Repeat the area calculation of Example 8 by projecting the part of the surface shown in Figure 15.23 onto the yz-plane and using the formula in Exercise 5(b).

7. Find $\iint_S x \, dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

8. Find the area of the part of the cone $z^2 = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 2ay$.

9. Find the area of the part of the cylinder $x^2 + y^2 = 2ay$ that lies outside the cone $z^2 = x^2 + y^2$.

10. Find the area of the part of the cylinder $x^2 + z^2 = a^2$ that lies inside the cylinder $y^2 + z^2 = a^2$.

11. A circular cylinder of radius a is circumscribed about a sphere of radius a so that the cylinder is tangent to the sphere along the equator. Two planes, each perpendicular to the axis of the cylinder, intersect the sphere and the cylinder in circles. Show that the area of that part of the sphere between the two planes is equal to the area of the part of the cylinder between the two planes. Thus, the area of the part of a sphere between two parallel planes that intersect it depends only on the radius of the sphere and the distance between the planes, and not on the particular position of the planes.

12. Let $0 < a < b$. In terms of the elliptic integral functions defined in Exercise 17 of Section 15.3, find the area of that part of each of the cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = b^2$ that lies inside the other cylinder.

13. Find $\iint_S y \, dS$, where S is the part of the plane $z = 1 + y$ that lies inside the cone $z = \sqrt{2(x^2 + y^2)}$.

14. Find $\iint_S y \, dS$, where S is the part of the cone $z = \sqrt{2(x^2 + y^2)}$ that lies below the plane $z = 1 + y$.

15. Find $\iint_S xz \, dS$, where S is the part of the surface $z = x^2$ that lies in the first octant of 3-space and inside the paraboloid $z = 1 - 3x^2 - y^2$.

16. Find the mass of the part of the surface $z = \sqrt{2xy}$ that lies above the region $0 \le x \le 5$, $0 \le y \le 2$, if the areal density of the surface is $\sigma(x, y, z) = kz$.

17. Find the total charge on the surface

$$\mathbf{r} = e^u \cos v \mathbf{i} + e^u \sin v \mathbf{j} + u\mathbf{k}, \quad (0 \le u \le 1,\ 0 \le v \le \pi),$$

if the charge density on the surface is $\delta = \sqrt{1 + e^{2u}}$.

Exercises 18–19 concern **spheroids**, which are ellipsoids with two of their three semi-axes equal, say $a = b$:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

*** 18.** Find the surface area of a **prolate spheroid**, where $0 < a < c$. A prolate spheroid has its two shorter semi-axes equal, like an (American) "pro football."

*** 19.** Find the surface area of an **oblate spheroid**, where $0 < c < a$. An oblate spheroid has its two longer semi-axes equal, like the earth.

20. Describe the parametric surface

$$x = au \cos v, \qquad y = au \sin v, \qquad z = bv,$$

$(0 \le u \le 1,\ 0 \le v \le 2\pi)$, and find its area.

*** 21.** Evaluate $\iint_{\mathcal{P}} \frac{dS}{(x^2 + y^2 + z^2)^{3/2}}$, where $\mathcal{P}$ is the plane with equation $Ax + By + Cz = D$, $(D \ne 0)$.

22. A spherical shell of radius a is centred at the origin. Find the centroid of that part of the sphere that lies in the first octant.

23. Find the centre of mass of a right-circular conical shell of base radius a, height h, and constant areal density σ.

*** 24.** Find the gravitational attraction of a hemispherical shell of radius a and constant areal density σ on a mass m located at the centre of the base of the hemisphere.

*** 25.** Find the gravitational attraction of a circular cylindrical shell of radius a, height h, and constant areal density σ on a mass m located on the axis of the cylinder b units above the base.

In Exercises 26–28, find the moment of inertia and radius of gyration of the given object about the given axis. Assume constant areal density σ in each case.

26. A cylindrical shell of radius a and height h about the axis of the cylinder

27. A spherical shell of radius a about a diameter

28. A right-circular conical shell of base radius a and height h about the a xis of the cone

29. With what acceleration will the spherical shell of Exercise 27 roll down a plane inclined at angle α to the horizontal? (Compare your result with that of Example 4(b) of Section 16.7.)

15.6 Oriented Surfaces and Flux Integrals

Surface integrals of normal components of vector fields play a very important role in vector calculus, similar to the role played by line integrals of tangential components of vector fields. Before we consider such surface integrals we need to define the *orientation* of a surface.

Oriented Surfaces

A smooth surface $\mathcal{S}$ in 3-space is said to be **orientable** if there exists a *unit vector field* $\hat{\mathbf{N}}(P)$ defined on $\mathcal{S}$, which varies continuously as P ranges over $\mathcal{S}$ and which is everywhere normal to $\mathcal{S}$. Any such vector field $\hat{\mathbf{N}}(P)$ determines an **orientation** of $\mathcal{S}$. The surface must have two sides since $\hat{\mathbf{N}}(P)$ can have only one value at each point P. The side out of which $\hat{\mathbf{N}}$ points is called the **positive side**; the other side is the **negative side**. An **oriented surface** is a smooth surface together with a particular choice of orienting unit normal vector field $\hat{\mathbf{N}}(P)$.

For example, if we define $\hat{\mathbf{N}}$ on the smooth surface $z = f(x, y)$ by

$$\hat{\mathbf{N}} = \frac{-f_1(x, y)\mathbf{i} - f_2(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{1 + (f_1(x, y))^2 + (f_2(x, y))^2}},$$

then the top of the surface is the positive side. (See Figure 15.26.)

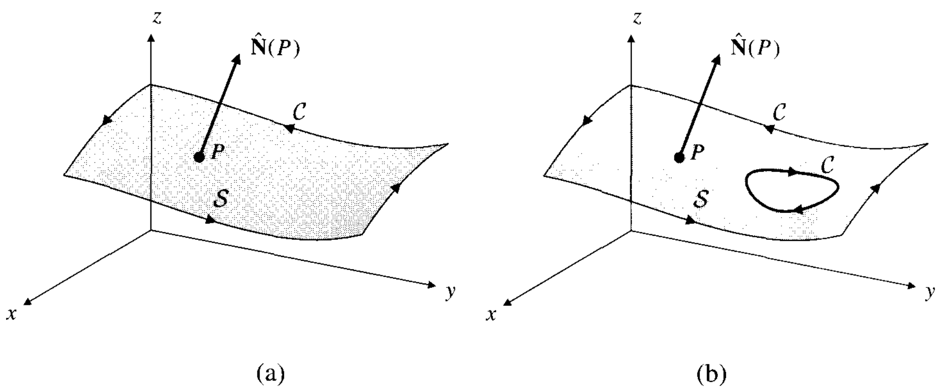

Figure 15.26 The boundary curves of an oriented surface are themselves oriented with the surface on the left

A smooth or piecewise smooth surface may be **closed** (i.e., it may have no boundary), or it may have one or more boundary curves. (The unit normal vector field $\hat{\mathbf{N}}(P)$ need not be defined at points of the boundary curves.)

An oriented surface $\mathcal{S}$ **induces an orientation** on any of its boundary curves $\mathcal{C}$; if we stand on the positive side of the surface $\mathcal{S}$ and walk around $\mathcal{C}$ in the direction of its orientation, then $\mathcal{S}$ will be on our left side. (See Figure 15.26(a) and (b).)

A *piecewise smooth* surface is **orientable** if, whenever two smooth component surfaces join along a common boundary curve $\mathcal{C}$, they induce *opposite* orientations along $\mathcal{C}$. This forces the normals $\hat{\mathbf{N}}$ to be on the same side of adjacent components. For instance, the surface of a cube is a piecewise smooth, closed surface, consisting of six smooth surfaces (the square faces) joined along edges. (See Figure 15.27.) If all of the faces are oriented so that their normals $\hat{\mathbf{N}}$ point out of the cube (or if they all point into the cube), then the surface of the cube itself is oriented.

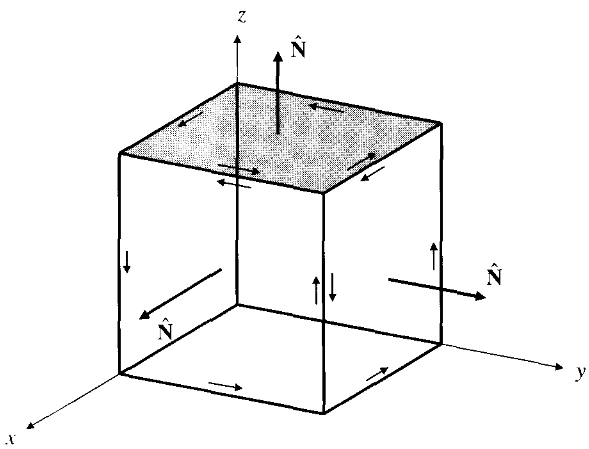

Figure 15.27 The surface of the cube is orientable; adjacent faces induce opposite orientations on their common edge

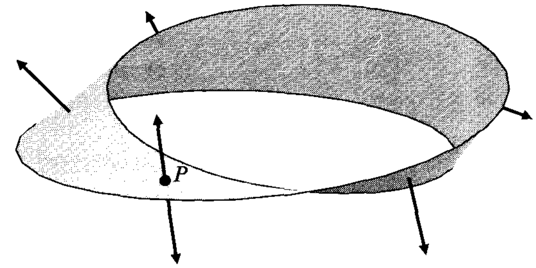

Figure 15.28 The Möbius band is not orientable; it has only one "side"

Not every surface can be oriented, even if it appears smooth. An orientable surface must have two sides. For example, a **Möbius band**, consisting of a strip of paper with ends joined together to form a loop, but with one end given a half twist before the ends are joined, has only one side (make one and see), so it cannot be oriented. (See Figure 15.28.) If a nonzero vector is moved around the band, starting at point P, so that it is always normal to the surface, then it can return to its starting position pointing in the opposite direction.

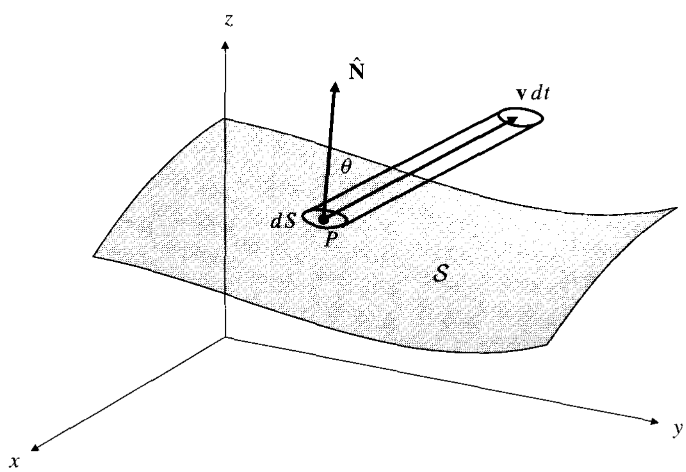

Figure 15.29 The fluid crossing dS in time dt fills the tube

The Flux of a Vector Field Across a Surface

Suppose 3-space is filled with an incompressible fluid that is flowing around with velocity field $\mathbf{v}$. Let $\mathcal{S}$ be an imaginary, smooth, oriented surface in 3-space. (We called $\mathcal{S}$ *imaginary* because it does not provide a barrier to the motion of the fluid. It is fixed in space, not moving with the fluid, and the fluid can move freely through it.) Let us calculate the rate at which fluid flows across $\mathcal{S}$. Let dS be a small area element at point P on the surface. The fluid crossing that element between time t and time $t + dt$ occupies a cylinder of base area dS and height $|\mathbf{v}(P)|\, dt\, \cos\theta$, where θ is the angle between $\mathbf{v}(P)$ and the normal $\hat{\mathbf{N}}(P)$. (See Figure 15.29.) This cylinder has (signed) volume $\mathbf{v}(P) \bullet \hat{\mathbf{N}}(P)\, dS\, dt$. The rate at which fluid is crossing dS is $\mathbf{v}(P) \bullet \hat{\mathbf{N}}(P)\, dS$, and the total rate at which it is crossing $\mathcal{S}$ is given by the surface integral

$$\iint_{\mathcal{S}} \mathbf{v} \bullet \hat{\mathbf{N}}\, dS \quad \text{or} \quad \iint_{\mathcal{S}} \mathbf{v} \bullet d\mathbf{S},$$

where we use $d\mathbf{S}$ to represent the vector surface area element $\hat{\mathbf{N}}\, dS$.

DEFINITION 6

Flux of a vector field across an oriented surface

Given any continuous vector field $\mathbf{F}$, the integral of the normal component of $\mathbf{F}$ over the oriented surface $\mathcal{S}$,

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \quad \text{or} \quad \iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S},$$

is called the **flux** of $\mathbf{F}$ across $\mathcal{S}$.

When the surface is closed, the flux integral can be denoted by

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \quad \text{or} \quad \oiint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}.$$

In this case we refer to the flux of $\mathbf{F}$ *out of* $\mathcal{S}$ if $\hat{\mathbf{N}}$ is the unit *exterior* normal, and the flux *into* $\mathcal{S}$ if $\hat{\mathbf{N}}$ is the unit *interior* normal.

Example 1 Find the flux of the vector field $\mathbf{F} = m\mathbf{r}/|\mathbf{r}|^3$ out of a sphere $\mathcal{S}$ of radius a centred at the origin. (Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.)

Solution Since $\mathbf{F}$ is the field associated with a source of strength m at the origin (which produces $4\pi m$ units of fluid per unit time at the origin), the answer must be $4\pi m$. Let us calculate it anyway. We use spherical coordinates. At any point $\mathbf{r}$ on the sphere, with spherical coordinates $[a, \phi, \theta]$, the unit outward normal is $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Since the vector field is $\mathbf{F} = m\hat{\mathbf{r}}/a^2$ on the sphere, and since an area element is $dS = a^2 \sin\phi\, d\phi\, d\theta$, the flux of $\mathbf{F}$ out of the sphere is

$$\oiint_{\mathcal{S}} \left(\frac{m}{a^2}\hat{\mathbf{r}}\right) \bullet \hat{\mathbf{r}}\, a^2 \sin\phi\, d\phi\, d\theta = m \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi = 4\pi m.$$

■

Example 2 Calculate the total flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the surface of the solid cylinder $x^2 + y^2 \le a^2$, $-h \le z \le h$.

Solution The cylinder is shown in Figure 15.30. Its surface consists of top and bottom disks and the cylindrical side wall. We calculate the flux of $\mathbf{F}$ out of each. Naturally, we use cylindrical coordinates.

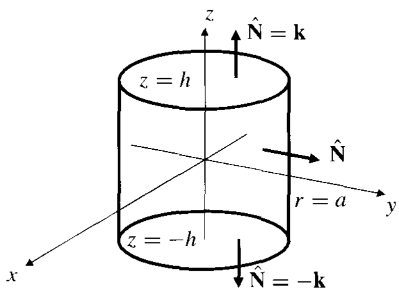

Figure 15.30 The three components of the surface of a solid cylinder with their outward normals

On the top disk we have $z = h$, $\hat{\mathbf{N}} = \mathbf{k}$, and $dS = r\,dr\,d\theta$. Therefore, $\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = hr\,dr\,d\theta$ and

$$\iint_{\text{top}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = h \int_0^{2\pi} d\theta \int_0^a r\,dr = \pi a^2 h.$$

On the bottom disk we have $z = -h$, $\hat{\mathbf{N}} = -\mathbf{k}$, and $dS = r\,dr\,d\theta$. Therefore, $\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = hr\,dr\,d\theta$ and

$$\iint_{\text{bottom}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iint_{\text{top}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \pi a^2 h.$$

On the cylindrical wall $\mathbf{F} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + z\mathbf{k}$, $\hat{\mathbf{N}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, and $dS = a\,d\theta\,dz$. Thus, $\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = a^2\,d\theta\,dz$ and

$$\iint_{\text{cylwall}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = a^2 \int_0^{2\pi} d\theta \int_{-h}^{h} dz = 4\pi a^2 h.$$

The total flux of $\mathbf{F}$ out of the surface $\mathcal{S}$ of the cylinder is the sum of these three contributions:

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = 6\pi a^2 h.$$

Let $\mathcal{S}$ be a smooth, oriented surface with a one-to-one projection onto a domain D in the xy-plane, and with equation of the form $G(x, y, z) = 0$. In Section 15.5 we showed that the surface area element on $\mathcal{S}$ could be written in the form

$$dS = \left|\frac{\nabla G}{G_3}\right| dx\,dy,$$

and hence surface integrals over $\mathcal{S}$ could be reduced to double integrals over the domain D. Flux integrals can be treated likewise. Depending on the orientation of $\mathcal{S}$, the unit normal $\hat{\mathbf{N}}$ can be written as

$$\hat{\mathbf{N}} = \pm \frac{\nabla G}{|\nabla G|}.$$

Thus, the vector area element $d\mathbf{S}$ can be written

$$d\mathbf{S} = \hat{\mathbf{N}}\,dS = \pm \frac{\nabla G(x, y, z)}{G_3(x, y, z)}\,dx\,dy.$$

The sign must be chosen to give $\mathcal{S}$ the desired orientation. If $G_3 > 0$ and we want the positive side of $\mathcal{S}$ to face upward, we should use the "+" sign. Of course, similar formulas apply for surfaces with one-to-one projections onto the other coordinate planes.

Example 3 Find the flux of $z\mathbf{i} + x^2\mathbf{k}$ upward through that part of the surface $z = x^2 + y^2$ lying above the square R defined by $-1 \le x \le 1$ and $-1 \le y \le 1$.

Solution For $F(x, y, z) = z - x^2 - y^2$ we have $\nabla F = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ and $F_3 = 1$. Thus,

$$d\mathbf{S} = (-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k})\, dx\, dy,$$

and the required flux is

$$\begin{aligned}\iint_S (z\mathbf{i} + x^2\mathbf{k}) \bullet d\mathbf{S} &= \iint_R \big(-2x(x^2 + y^2) + x^2\big)\, dx\, dy \\ &= \int_{-1}^{1} dx \int_{-1}^{1} (x^2 - 2x^3 - 2xy^2)\, dy \\ &= \int_{-1}^{1} 2x^2\, dx = \frac{4}{3}.\end{aligned}$$

(Two of the three terms in the double integral had zero integrals because of symmetry.) ■

For a surface $\mathcal{S}$ with equation $z = f(x, y)$ we have

$$\hat{\mathbf{N}} = \pm \frac{-\dfrac{\partial f}{\partial x}\mathbf{i} - \dfrac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}} \quad \text{and}$$

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy,$$

so that the vector area element on $\mathcal{S}$ is given by

$$d\mathbf{S} = \hat{\mathbf{N}}\, dS = \pm\left(-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}\right) dx\, dy.$$

Again, the $+$ sign corresponds to an upward normal.

For a general parametric surface $\mathbf{r} = \mathbf{r}(u, v)$, the unit normal $\hat{\mathbf{N}}$ and area element dS were calculated in Section 15.5:

$$\hat{\mathbf{N}} = \pm \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left|\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}\right|}, \qquad dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv.$$

Thus, the vector area element is

$$d\mathbf{S} = \hat{\mathbf{N}}\, dS = \pm\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du\, dv.$$

Example 4 Find the flux of $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 4\mathbf{k}$ upward through $\mathcal{S}$, where $\mathcal{S}$ is the part of the surface $z = 1 - x^2 - y^2$ lying in the first octant of 3-space.

Solution The vector area element corresponding to the upward normal on $\mathcal{S}$ is

$$d\mathbf{S} = \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dx\,dy = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dx\,dy.$$

The projection of $\mathcal{S}$ onto the xy-plane is the quarter-circular disk Q given by $x^2 + y^2 \le 1$, $x \ge 0$, and $y \ge 0$. Thus, the flux of $\mathbf{F}$ upward through $\mathcal{S}$ is

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} &= \iint_{Q} (2xy - 2xy + 4)\,dx\,dy \\ &= 4 \times (\text{area of } Q) = \pi.\end{aligned}$$

■

Example 5 Find the flux of

$$\mathbf{F} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{x^2 + y^2} + \mathbf{k}$$

downward through the surface $\mathcal{S}$ defined parametrically by

$$\mathbf{r} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k}, \qquad (0 \le u \le 1,\ 0 \le v \le 2\pi).$$

Solution First we calculate $d\mathbf{S}$:

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= \cos v\mathbf{i} + \sin v\mathbf{j} + 2u\mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial v} &= -u\sin v\mathbf{i} + u\cos v\mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= -2u^2\cos v\mathbf{i} - 2u^2\sin v\mathbf{j} + u\mathbf{k}.\end{aligned}$$

Since $u \ge 0$ on $\mathcal{S}$, the latter expression is an upward normal. We want a downward normal, so we use

$$d\mathbf{S} = (2u^2\cos v\mathbf{i} + 2u^2\sin v\mathbf{j} - u\mathbf{k})\,du\,dv.$$

On $\mathcal{S}$ we have

$$\mathbf{F} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{x^2 + y^2} + \mathbf{k} = \frac{2u\cos v\mathbf{i} + 2u\sin v\mathbf{j}}{u^2} + \mathbf{k},$$

so the downward flux of $\mathbf{F}$ through $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \int_0^{2\pi} dv \int_0^1 (4u - u)\,du = 3\pi.$$

■

Exercises 15.6

1. Find the flux of $\mathbf{F} = x\mathbf{i} + z\mathbf{j}$ out of the tetrahedron bounded by the coordinate planes and the plane $x + 2y + 3z = 6$.
2. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward across the sphere $x^2 + y^2 + z^2 = a^2$.
3. Find the flux of the vector field of Exercise 2 out of the surface of the rectangular box $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.
4. Find the flux of the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{k}$ out across the boundary of the solid cone $0 \le z \le 1 - \sqrt{x^2 + y^2}$.
5. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through the part of the surface $z = a - x^2 - y^2$ lying above the plane $z = b$, where $b < a$.
6. Find the flux of $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + \mathbf{k}$ upward through the part of the surface $z = x^2 - y^2$ lying inside the cylinder $x^2 + y^2 = a^2$.
7. Find the flux of $\mathbf{F} = y^3\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$ downward through the part of the surface $z = 4 - x^2 - y^2$ that lies above the plane $z = 2x + 1$.
8. Find the flux of $\mathbf{F} = z^2\mathbf{k}$ upward through the part of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant of 3-space.
9. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ upward through the part of the surface $z = 2 - x^2 - 2y^2$ that lies above the xy-plane.
10. Find the flux of $\mathbf{F} = 2x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through the surface $\mathbf{r} = u^2v\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}$, $(0 \le u \le 1, 0 \le v \le 1)$.
11. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ upward through the surface $u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$, $(0 \le u \le 2, 0 \le v \le \pi)$.
12. Find the flux of $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + (x^2 + y^2)\mathbf{k}$ upward through the surface $\mathbf{r} = e^u\cos v\,\mathbf{i} + e^u\sin v\,\mathbf{j} + u\,\mathbf{k}$, where $0 \le u \le 1$ and $0 \le v \le \pi$.
13. Find the flux of $\mathbf{F} = m\mathbf{r}/|\mathbf{r}|^3$ out of the surface of the cube $-a \le x, y, z \le a$.

* 14. Find the flux of the vector field of Exercise 13 out of the box $1 \le x, y, z \le 2$. *Note:* This problem can be solved very easily using the Divergence Theorem of Section 16.4; the required flux is, in fact, zero. However, the object here is to do it by direct calculation of the surface integrals involved, and as such it is quite difficult. By symmetry, it is sufficient to evaluate the net flux out of the cube through any one of the three pairs of opposite faces; that is, you must calculate the flux through only two faces, say $z = 1$ and $z = 2$. Be prepared to work very hard to evaluate these integrals! When they are done you may find the identities

$$2\arctan a = \arctan\left(\frac{2a}{1-a^2}\right) \quad \text{and}$$

$$\arctan a + \arctan\left(\frac{1}{a}\right) = \frac{\pi}{2}$$

useful for showing that the net flux is zero.

15. Define the flux of a *plane* vector field across a piecewise smooth *curve*. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ outward across
    (a) the circle $x^2 + y^2 = a^2$ and
    (b) the boundary of the square $-1 \le x, y \le 1$.
16. Find the flux of $\mathbf{F} = -\dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$ *inward* across each of the two curves in the previous exercise.
17. If $\mathcal{S}$ is a smooth, oriented surface in 3-space and $\hat{\mathbf{N}}$ is the unit vector field determining the orientation of $\mathcal{S}$, show that the flux of $\hat{\mathbf{N}}$ across $\mathcal{S}$ is the area of $\mathcal{S}$.

* 18. The Divergence Theorem presented in Section 16.4 implies that the flux of a constant vector field across any oriented, piecewise smooth, closed surface is zero. Prove this now for (a) a rectangular box and (b) a sphere.

Chapter Review

Key Ideas

- **What do the following terms and phrases mean?**
  - ◇ a vector field ◇ a scalar field
  - ◇ a field line ◇ a conservative field
  - ◇ a scalar potential ◇ an equipotential
  - ◇ a source ◇ a dipole
  - ◇ a connected domain ◇ a simply connected domain
  - ◇ a parametric surface ◇ an orientable surface
  - ◇ the line integral of f along curve $\mathcal{C}$
  - ◇ the line integral of the tangential component of $\mathbf{F}$ along $\mathcal{C}$
  - ◇ the flux of a vector field through a surface
- **How are the field lines of a conservative field related to its equipotential curves or surfaces?**
- **How is a line integral of a scalar field calculated?**
- **How is a line integral of the tangential component of a vector field calculated?**
- **When is a line integral between two points independent of the path joining those points?**
- **How is a surface integral of a scalar field calculated?**
- **How is the flux of a vector field through a surface calculated?**

Review Exercises

1. Find $\int_C \frac{1}{y}\,ds$, where C is the curve

$$x = t, \quad y = 2e^t, \quad z = e^{2t}, \quad (-1 \le t \le 1).$$

2. Let C be the part of the curve of intersection of the surfaces $z = x + y^2$ and $y = 2x$ from the origin to the point $(2, 4, 18)$. Evaluate $\int_C 2y\,dx + x\,dy + 2\,dz$.

3. Find $\iint_S x\,dS$, where S is that part of the cone $z = \sqrt{x^2 + y^2}$ in the region $0 \le x \le 1 - y^2$.

4. Find $\iint_S xyz\,dS$ over the part of the plane $x + y + z = 1$ lying in the first octant.

5. Find the flux of $x^2y\mathbf{i} - 10xy^2\mathbf{j}$ upward through the surface $z = xy$, $0 \le x \le 1$, $0 \le y \le 1$.

6. Find the flux of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ downward through the part of the plane $x + 2y + 3z = 6$ lying in the first octant.

7. A bead of mass m slides down a wire in the shape of the curve

$$x = a\sin t, \quad y = a\cos t, \quad z = bt,$$

where $0 \le t \le 6\pi$.

(a) What is the work done by the gravitational force $\mathbf{F} = -mg\mathbf{k}$ on the bead during its descent?

(b) What is the work done against a resistance of constant magnitude R which directly opposes the motion of the bead during its descent?

8. For what values of the constants a, b, and c can you determine the value of the integral I of the tangential component of

$$\mathbf{F} = (axy + 3yz)\mathbf{i} + (x^2 + 3xz + by^2z)\mathbf{j} + (bxy + cy^3)\mathbf{k}$$

along a curve from $(0, 1, -1)$ to $(2, 1, 1)$ without knowing exactly which curve? What is the value of the integral?

9. Let $\mathbf{F} = (x^2/y)\mathbf{i} + y\mathbf{j} + \mathbf{k}$.

(a) Find the field line of $\mathbf{F}$ that passes through $(1, 1, 0)$ and show that it also passes through $(e, e, 1)$.

(b) Find $\int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the part of the field line in (a) from $(1, 1, 0)$ to $(e, e, 1)$.

10. Consider the vector fields

$$\mathbf{F} = (1 + x)e^{x+y}\mathbf{i} + (xe^{x+y} + 2y)\mathbf{j} - 2z\mathbf{k},$$
$$\mathbf{G} = (1 + x)e^{x+y}\mathbf{i} + (xe^{x+y} + 2z)\mathbf{j} - 2y\mathbf{k}.$$

(a) Show that $\mathbf{F}$ is conservative by finding a potential for it.

(b) Evaluate $\int_C \mathbf{G} \bullet d\mathbf{r}$, where C is given by

$$\mathbf{r} = (1 - t)e^t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, \quad (0 \le t \le 1),$$

by taking advantage of the similarity between $\mathbf{F}$ and $\mathbf{G}$.

11. Find a plane vector field $\mathbf{F}(x, y)$ that satisfies all of the following conditions:

(i) The field lines of $\mathbf{F}$ are the curves $xy = C$.

(ii) $|\mathbf{F}(x, y)| = 1$ if $(x, y) \ne (0, 0)$.

(iii) $\mathbf{F}(1, 1) = (\mathbf{i} - \mathbf{j})/\sqrt{2}$.

(iv) $\mathbf{F}$ is continuous except at $(0, 0)$.

12. Let S be the part of the surface of the cylinder $y^2 + z^2 = 16$ that lies in the first octant and between the planes $x = 0$ and $x = 5$. Find the flux of $3z^2x\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ away from the x-axis through S.

Challenging Problems

* 1. Find the centroid of the surface

$$\mathbf{r} = (2 + \cos v)(\cos u\mathbf{i} + \sin u\mathbf{j}) + \sin v\mathbf{k},$$

where $0 \le u \le 2\pi$ and $0 \le v \le \pi$. Describe this surface.

* 2. A smooth surface S is given parametrically by

$$\mathbf{r} = (\cos 2u)(2 + v\cos u)\mathbf{i} + (\sin 2u)(2 + v\cos u)\mathbf{j} + v\sin u\mathbf{k},$$

where $0 \le u \le 2\pi$ and $-1 \le v \le 1$. Show that for *every* smooth vector field $\mathbf{F}$ on S,

$$\iint_S \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = 0,$$

where $\hat{\mathbf{N}} = \hat{\mathbf{N}}(u, v)$ is a unit normal vector field on S that depends continuously on (u, v). How do you explain this? *Hint:* try to describe what the surface S looks like.

* 3. Recalculate the gravitational force exerted by a sphere of radius a and areal density σ centred at the origin on a point mass located at $(0, 0, b)$ by directly integrating the vertical component of the force due to an area element dS, rather than by integrating the potential as we did in the last part of Section 15.5. You will have to be quite creative in dealing with the resulting integral.