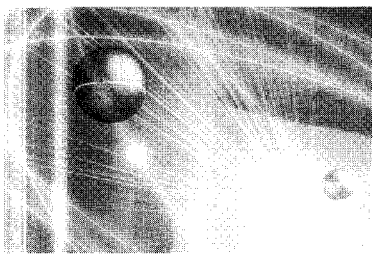

CHAPTER 9

Sequences, Series, and Power Series

Introduction An infinite series is a sum that involves infinitely many terms. Since addition is carried out on two numbers at a time, the evaluation of the sum of an infinite series necessarily involves finding a limit. Complicated functions $f(x)$ can frequently be expressed as series of simpler functions. For example, many of the transcendental functions we have encountered can be expressed as series of powers of x so that they resemble polynomials of infinite degree. Such series can be differentiated and integrated term by term, and they play a very important role in the study of calculus.

9.1 Sequences and Convergence

By a **sequence** (or an **infinite sequence**) we mean an ordered list having a first element but no last element. For our purposes, the elements (called **terms**) of a sequence will always be real numbers, although much of our discussion could be applied to complex numbers as well. Examples of sequences are:

$\{1, 2, 3, 4, 5, \ldots\}$ the sequence of positive integers,

$\left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots\right\}$ the sequence of positive integer powers of $-\frac{1}{2}$.

The terms of a sequence are usually listed in braces { } as shown. The ellipsis (. . .) should be read "and so on."

An infinite sequence is a special kind of function, one whose domain is a set of integers extending from some starting integer to infinity. The starting integer is usually 1, so the domain is the set of positive integers. The sequence $\{a_1, a_2, a_3, a_4, \ldots\}$ is the function f that takes the value $f(n) = a_n$ at each positive integer n. A sequence can be specified in three ways:

(i) We can list the first few terms followed by . . . *if the pattern is obvious.*

(ii) We can provide a formula for the **general term** a_n as a function of n.

(iii) We can provide a formula for calculating the term a_n as a function of earlier terms $a_1, a_2, \ldots, a_{n-1}$ and specify enough of the beginning terms so the process of computing higher terms can begin.

In each case it must be possible to determine any term of the sequence, although it may be necessary to calculate all the preceding terms first.

Example 1 **(Some examples of sequences)**

(a) $\{n\} = \{1, 2, 3, 4, 5, \ldots\}$

(b) $\left\{\left(-\frac{1}{2}\right)^n\right\} = \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots\right\}$

(c) $\left\{\dfrac{n-1}{n}\right\} = \left\{0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots\right\}$

(d) $\{(-1)^{n-1}\} = \{\cos((n-1)\pi)\} = \{1, -1, 1, -1, 1, \ldots\}$

(e) $\left\{\dfrac{n^2}{2^n}\right\} = \left\{\dfrac{1}{2}, 1, \dfrac{9}{8}, 1, \dfrac{25}{32}, \dfrac{36}{64}, \dfrac{49}{128}, \ldots\right\}$

(f) $\left\{\left(1+\dfrac{1}{n}\right)^n\right\} = \left\{2, \left(\dfrac{3}{2}\right)^2, \left(\dfrac{4}{3}\right)^3, \left(\dfrac{5}{4}\right)^4, \ldots\right\}$

(g) $\left\{\dfrac{\cos(n\pi/2)}{n}\right\} = \left\{0, -\dfrac{1}{2}, 0, \dfrac{1}{4}, 0, -\dfrac{1}{6}, 0, \dfrac{1}{8}, 0, \ldots\right\}$

(h) $a_1 = 1, \quad a_{n+1} = \sqrt{6 + a_n}, \quad (n = 1, 2, 3, \ldots)$
In this case $\{a_n\} = \{1, \sqrt{7}, \sqrt{6+\sqrt{7}}, \ldots\}$.
Note that there is no *obvious* formula for a_n as an explicit function of n here, but we can still calculate a_n for any desired value of n provided we first calculate all the earlier values $a_2, a_3, \ldots, a_{n-1}$.

(i) $a_1 = 1, a_2 = 1, a_{n+2} = a_n + a_{n+1}, \quad (n = 1, 2, 3, \ldots)$
Here $\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$.
This is called the **Fibonacci sequence**. Each term after the second is the sum of the previous two terms. ■

In parts (a)–(g) of Example 1, the formulas on the left sides define the general term of each sequence $\{a_n\}$ as an explicit function of n. In parts (h) and (i) we say the sequence $\{a_n\}$ is defined **recursively** or **inductively**; each term must be calculated from previous ones rather than directly as a function of n.

The following definition introduces terminology used to describe various properties of sequences.

DEFINITION 1

Terms for describing sequences

(a) The sequence $\{a_n\}$ is **bounded below** by L, and L is a **lower bound** for $\{a_n\}$, if $a_n \geq L$ for every $n = 1, 2, 3, \ldots$. The sequence is **bounded above** by M, and M is an **upper bound**, if $a_n \leq M$ for every such n.

The sequence $\{a_n\}$ is **bounded** if it is both bounded above and bounded below. In this case there is a constant K such that $|a_n| \leq K$ for every $n = 1, 2, 3, \ldots$. (We can take K to be the larger of $-L$ and M.)

(b) The sequence $\{a_n\}$ is **positive** if it is bounded below by zero, that is, if $a_n \geq 0$ for every $n = 1, 2, 3, \ldots$; it is **negative** if $a_n \leq 0$ for every n.

(c) The sequence $\{a_n\}$ is **increasing** if $a_{n+1} \geq a_n$ for every $n = 1, 2, 3, \ldots$; it is **decreasing** if $a_{n+1} \leq a_n$ for every such n. The sequence is said to be **monotonic** if it is either increasing or decreasing. (The terminology here is looser than that we have used for functions, where we would have used *nondecreasing* and *nonincreasing* to describe this behaviour.)

(d) The sequence $\{a_n\}$ is **alternating** if $a_n a_{n+1} < 0$ for every $n = 1, 2, \ldots$, that is, if any two consecutive terms have opposite sign. Note that this definition requires $a_n \neq 0$ for each n.

Example 2 **(Describing some sequences)**

(a) The sequence $\{n\} = \{1, 2, 3, \ldots\}$ is positive, increasing, and bounded below. A lower bound for the sequence is 1 or any smaller number. The sequence is not bounded above.

(b) $\left\{\dfrac{n-1}{n}\right\} = \left\{0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots\right\}$ is positive, bounded, and increasing. Here 0 is a lower bound and 1 is an upper bound.

(c) $\left\{\left(-\dfrac{1}{2}\right)^n\right\} = \left\{-\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \dfrac{1}{16}, \ldots\right\}$ is bounded and alternating. Here $-1/2$ is a lower bound and $1/4$ is an upper bound.

(d) $\{(-1)^n n\} = \{-1, 2, -3, 4, -5, \ldots\}$ is alternating but not bounded either above or below. ■

When you want to show that a sequence is increasing, you can try to show that the inequality $a_{n+1} - a_n \geq 0$ holds for $n \geq 1$. Alternatively, if $a_n = f(n)$ for a differentiable function $f(x)$, you can show that f is a nondecreasing function on $[1, \infty[$ by showing that $f'(x) \geq 0$ there. Similar approaches are useful for showing that a sequence is decreasing.

Example 3 If $a_n = \dfrac{n}{n^2+1}$, show that the sequence $\{a_n\}$ is decreasing.

Solution Since $a_n = f(n)$, where $f(x) = \dfrac{x}{x^2+1}$ and

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0 \quad \text{for } x \geq 1,$$

the function $f(x)$ is decreasing on $[1, \infty[$; therefore, $\{a_n\}$ is a decreasing sequence. ■

The sequence $\left\{\dfrac{n^2}{2^n}\right\} = \left\{\dfrac{1}{2}, 1, \dfrac{9}{8}, 1, \dfrac{25}{32}, \dfrac{36}{64}, \dfrac{49}{128}, \ldots\right\}$ is positive and therefore bounded below. It seems clear that from the fourth term on, all the terms are getting smaller. However, $a_2 > a_1$ and $a_3 > a_2$. Since $a_{n+1} \leq a_n$ only if $n \geq 3$, we say that this sequence is **ultimately decreasing**. The adverb *ultimately* is used to describe any termwise property of a sequence that the terms have from some point on, but not necessarily at the beginning of the sequence. Thus, the sequence

$$\{n - 100\} = \{-99, -98, \ldots, -2, -1, 0, 1, 2, 3, \ldots\}$$

is *ultimately positive* even though the first 99 terms are negative, and the sequence

$$\left\{(-1)^n + \frac{4}{n}\right\} = \left\{3, 3, \frac{1}{3}, 2, -\frac{1}{5}, \frac{5}{3}, -\frac{3}{7}, \frac{3}{2}, \ldots\right\}$$

is *ultimately alternating* even though the first few terms do not alternate.

Convergence of Sequences

Central to the study of sequences is the notion of convergence. The concept of the limit of a sequence is a special case of the concept of the limit of a function $f(x)$

as $x \to \infty$. We say that the sequence $\{a_n\}$ **converges to the limit** L, and we write $\lim_{n\to\infty} a_n = L$, provided the distance from a_n to L on the real line approaches 0 as n increases toward ∞. We state this definition more formally as follows:

DEFINITION 2

Limit of a sequence

We say that sequence $\{a_n\}$ converges to the limit L, and we write $\lim_{n\to\infty} a_n = L$, if for every positive real number ϵ there exists an integer N (which may depend on ϵ) such that if $n \geq N$, then $|a_n - L| < \epsilon$.

This definition is illustrated in Figure 9.1.

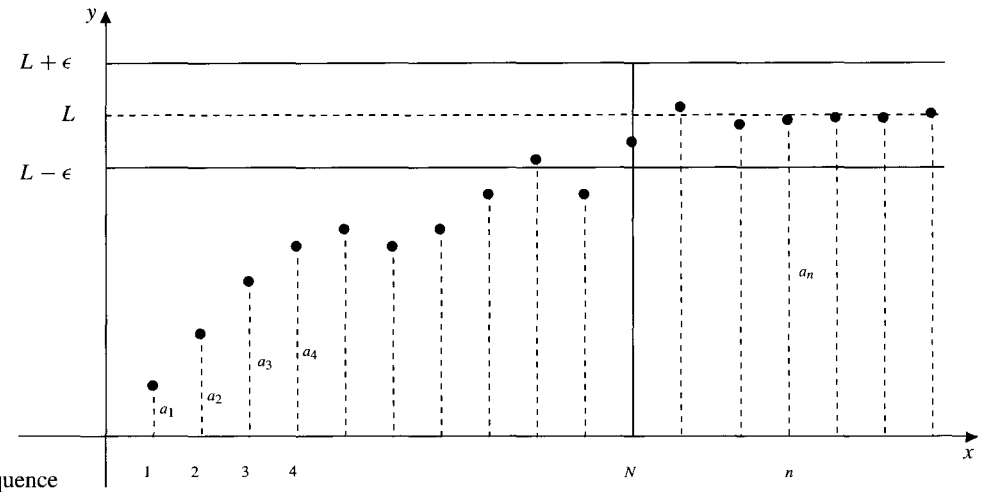

Figure 9.1 A convergent sequence

Example 4 Show that $\lim_{n\to\infty} \dfrac{c}{n^p} = 0$ for any real number c and any $p > 0$.

Solution Let $\epsilon > 0$ be given. Then

$$\left|\frac{c}{n^p}\right| < \epsilon \quad \text{if} \quad n^p > \frac{|c|}{\epsilon},$$

that is, if $n \geq N$, the least integer greater than $(|c|/\epsilon)^{1/p}$. By Definition 2, $\lim_{n\to\infty} \dfrac{c}{n^p} = 0$.

Every sequence $\{a_n\}$ must either **converge** to a finite limit L or **diverge**. That is, either $\lim_{n\to\infty} a_n = L$ exists (is a real number) or $\lim_{n\to\infty} a_n$ does not exist. If $\lim_{n\to\infty} a_n = \infty$, we can say that the sequence diverges to ∞; if $\lim_{n\to\infty} a_n = -\infty$, we can say that it diverges to $-\infty$. If $\lim_{n\to\infty} a_n$ simply does not exist (but is not ∞ or $-\infty$), we can only say that the sequence diverges.

Example 5 **(Examples of convergent and divergent sequences)**

(a) $\{(n-1)/n\}$ converges to 1; $\lim_{n\to\infty}(n-1)/n = \lim_{n\to\infty}\big(1 - (1/n)\big) = 1$.

(b) $\{n\} = \{1, 2, 3, 4, \ldots\}$ diverges to ∞.

(c) $\{-n\} = \{-1, -2, -3, -4, \ldots\}$ diverges to $-\infty$.

(d) $\{(-1)^n\} = \{-1, 1, -1, 1, -1, \ldots\}$ simply diverges.

(e) $\{(-1)^n n\} = \{-1, 2, -3, 4, -5, \ldots\}$ diverges (but not to ∞ or $-\infty$ even though $\lim_{n\to\infty} |a_n| = \infty$). ■

The limit of a sequence is equivalent to the limit of a function as its argument approaches infinity:

$$\text{If } \lim_{x\to\infty} f(x) = L \text{ and } a_n = f(n), \text{ then } \lim_{n\to\infty} a_n = L.$$

Because of this, the standard rules for limits of functions (Theorems 2 and 4 of Section 1.2) also hold for limits of sequences, with the appropriate changes of notation. Thus, if $\{a_n\}$ and $\{b_n\}$ converge, then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n,$$

$$\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n),$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{assuming } \lim_{n\to\infty} b_n \neq 0.$$

$$\text{If } a_n \leq b_n \text{ ultimately, then } \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

$$\text{If } a_n \leq b_n \leq c_n \text{ ultimately, and } \lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n, \text{ then } \lim_{n\to\infty} b_n = L.$$

The limits of many explicitly defined sequences can be evaluated using these properties in a manner similar to the methods used for limits of the form $\lim_{x\to\infty} f(x)$ in Section 1.3.

Example 6 Calculate the limits of the sequences

$$\text{(a) } \left\{\frac{2n^2 - n - 1}{5n^2 + n - 3}\right\}, \quad \text{(b) } \left\{\frac{\cos n}{n}\right\}, \quad \text{and} \quad \text{(c) } \{\sqrt{n^2 + 2n} - n\}.$$

Solution

(a) We divide the numerator and denominator of the expression for a_n by the highest power of n in the denominator, that is, by n^2:

$$\lim_{n\to\infty} \frac{2n^2 - n - 1}{5n^2 + n - 3} = \lim_{n\to\infty} \frac{2 - (1/n) - (1/n^2)}{5 + (1/n) - (3/n^2)} = \frac{2 - 0 - 0}{5 + 0 - 0} = \frac{2}{5},$$

since $\lim_{n\to\infty} 1/n = 0$ and $\lim_{n\to\infty} 1/n^2 = 0$. The sequence converges and its limit is 2/5.

(b) Since $|\cos n| \leq 1$ for every n, we have

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n} \quad \text{for} \quad n \geq 1.$$

Now, $\lim_{n\to\infty} -1/n = 0$ and $\lim_{n\to\infty} 1/n = 0$. Therefore, by the sequence version of the Squeeze Theorem, $\lim_{n\to\infty} (\cos n)/n = 0$. The given sequence converges to 0.

(c) For this sequence we multiply the numerator and the denominator (which is 1) by the conjugate of the expression in the numerator:

The sequence converges to 1. ■

Example 7 Evaluate $\lim_{n\to\infty} n \tan^{-1}\left(\frac{1}{n}\right)$.

Solution For this example it is best to replace the nth term of the sequence by the corresponding function of a real variable x and take the limit as $x \to \infty$. We use l'Hôpital's Rule:

$$\begin{aligned}
\lim_{n\to\infty} n \tan^{-1}\left(\frac{1}{n}\right) &= \lim_{x\to\infty} x \tan^{-1}\left(\frac{1}{x}\right) \\
&= \lim_{x\to\infty} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}} \qquad \left[\frac{0}{0}\right] \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+(1/x^2)}\left(-\frac{1}{x^2}\right)}{-\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1.
\end{aligned}$$

■

THEOREM 1 If $\{a_n\}$ converges, then $\{a_n\}$ is bounded.

PROOF Suppose $\lim_{n\to\infty} a_n = L$. According to Definition 2, for $\epsilon = 1$ there exists a number N such that if $n > N$, then $|a_n - L| < 1$; therefore $|a_n| < 1 + |L|$ for such n. (Why is this true?) If K denotes the largest of the numbers $|a_1|$, $|a_2|$, $\ldots$, $|a_N|$, and $1 + |L|$, then $|a_n| \le K$ for every $n = 1, 2, 3, \ldots$. Hence $\{a_n\}$ is bounded.

The converse of Theorem 1 is false; the sequence $\{(-1)^n\}$ is bounded but does not converge.

The *completeness property* of the real number system (see Section P.1) can be reformulated in terms of sequences to read as follows:

Bounded monotonic sequences converge

If the sequence $\{a_n\}$ is bounded above and is (ultimately) increasing, then it converges. The same conclusion holds if $\{a_n\}$ is bounded below and is (ultimately) decreasing.

Thus, a bounded, ultimately monotonic sequence is convergent. (See Figure 9.2.)

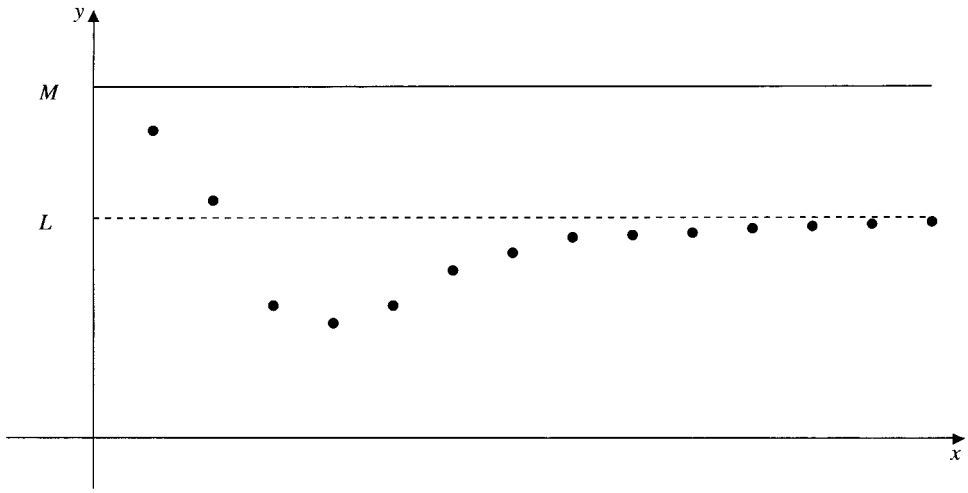

Figure 9.2 An ultimately increasing sequence that is bounded above

Example 8 Let a_n be defined recursively by

$$a_1 = 1, \qquad a_{n+1} = \sqrt{6 + a_n} \qquad (n = 1, 2, 3, \ldots).$$

Show that $\lim_{n\to\infty} a_n$ exists and find its value.

Solution Observe that $a_2 = \sqrt{6+1} = \sqrt{7} > a_1$. If $a_{k+1} > a_k$, then $a_{k+2} = \sqrt{6 + a_{k+1}} > \sqrt{6 + a_k} = a_{k+1}$, so $\{a_n\}$ is increasing, by induction. Now observe that $a_1 = 1 < 3$. If $a_k < 3$, then $a_{k+1} = \sqrt{6 + a_k} < \sqrt{6+3} = 3$, so $a_n < 3$ for every n by induction. Since $\{a_n\}$ is increasing and bounded above, $\lim_{n\to\infty} a_n = a$ exists, by completeness. Since $\sqrt{6+x}$ is a continuous function of x, we have

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + \lim_{n\to\infty} a_n} = \sqrt{6 + a}.$$

Thus $a^2 = 6 + a$, or $a^2 - a - 6 = 0$, or $(a-3)(a+2) = 0$. This quadratic has roots $a = 3$ and $a = -2$. Since $a_n \geq 1$ for every n, we must have $a \geq 1$. Therefore, $a = 3$ and $\lim_{n\to\infty} a_n = 3$. ■

There is a subtle point to note in this solution. Showing that $\{a_n\}$ is increasing is pretty obvious, but how did we know to try and show that 3 (rather than some other number) was an upper bound? The answer is that we actually did the last part first and showed that *if* $\lim a_n = a$ exists, *then* $a = 3$. It then makes sense to try and show that $a_n < 3$ for all n.

Example 9 Does $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ converge or not?

Solution We could make an effort to show that the given sequence is, in fact, increasing and bounded above. (See Exercise 32 at the end of this section.) However, we already know the answer. By Theorem 6 of Section 3.4,

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e^1 = e.$$

■

THEOREM 2 If $\{a_n\}$ is (ultimately) increasing, then either it is bounded above, and therefore convergent, or it is not bounded above and diverges to infinity.

The proof of this theorem is left as an exercise. A corresponding result holds for (ultimately) decreasing sequences.

The following theorem evaluates two important limits that find frequent application in the study of series.

THEOREM 3

(a) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

(b) If x is any real number, then $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

PROOF For part (a) observe that

$$\lim_{n\to\infty} \ln|x|^n = \lim_{n\to\infty} n\ln|x| = -\infty,$$

since $\ln|x| < 0$ when $|x| < 1$. Accordingly, since e^x is continuous,

$$\lim_{n\to\infty} |x|^n = \lim_{n\to\infty} e^{\ln|x|^n} = e^{\lim_{n\to\infty}\ln|x|^n} = 0.$$

Since $-|x|^n \le x^n \le |x|^n$, we have $\lim_{n\to\infty} x^n = 0$ by the Squeeze Theorem.

For part (b), pick any x and let N be an integer such that $N > |x|$. If $n > N$ we have

$$\begin{aligned}\left|\frac{x^n}{n!}\right| &= \frac{|x|}{1}\frac{|x|}{2}\frac{|x|}{3}\cdots\frac{|x|}{N-1}\frac{|x|}{N}\frac{|x|}{N+1}\cdots\frac{|x|}{n}\\ &< \frac{|x|^{N-1}}{(N-1)!}\frac{|x|}{N}\frac{|x|}{N}\frac{|x|}{N}\cdots\frac{|x|}{N}\\ &= \frac{|x|^{N-1}}{(N-1)!}\left(\frac{|x|}{N}\right)^{n-N+1} = K\left(\frac{|x|}{N}\right)^n,\end{aligned}$$

where $K = \frac{|x|^{N-1}}{(N-1)!}\left(\frac{|x|}{N}\right)^{1-N}$ is a constant that is independent of n. Since $|x|/N < 1$, we have $\lim_{n\to\infty}(|x|/N)^n = 0$ by part (a). Thus $\lim_{n\to\infty}|x^n/n!| = 0$, so $\lim_{n\to\infty} x^n/n! = 0$.

Example 10 Find $\lim_{n\to\infty}\frac{3^n+4^n+5^n}{5^n}$.

Solution $\lim_{n\to\infty}\frac{3^n+4^n+5^n}{5^n} = \lim_{n\to\infty}\left[\left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n + 1\right] = 0+0+1 = 1$, by Theorem 3(a).

Exercises 9.1

In Exercises 1–13, determine whether the given sequence is (a) bounded (above or below), (b) positive or negative (ultimately), (c) increasing, decreasing, or alternating, and (d) convergent, divergent, divergent to ∞ or $-\infty$.

1. $\left\{\frac{2n^2}{n^2+1}\right\}$

2. $\left\{\frac{2n}{n^2+1}\right\}$

3. $\left\{4 - \frac{(-1)^n}{n}\right\}$

4. $\left\{\sin\frac{1}{n}\right\}$

5. $\left\{\dfrac{n^2-1}{n}\right\}$

6. $\left\{\dfrac{e^n}{\pi^n}\right\}$

7. $\left\{\dfrac{e^n}{\pi^{n/2}}\right\}$

8. $\left\{\dfrac{(-1)^n n}{e^n}\right\}$

9. $\left\{\dfrac{2^n}{n^n}\right\}$

10. $\left\{\dfrac{(n!)^2}{(2n)!}\right\}$

11. $\left\{n\cos\left(\dfrac{n\pi}{2}\right)\right\}$

12. $\left\{\dfrac{\sin n}{n}\right\}$

13. $\{1, 1, -2, 3, 3, -4, 5, 5, -6, \ldots\}$

In Exercises 14–29, evaluate, wherever possible, the limit of the sequence $\{a_n\}$.

14. $a_n = \dfrac{5-2n}{3n-7}$

15. $a_n = \dfrac{n^2-4}{n+5}$

16. $a_n = \dfrac{n^2}{n^3+1}$

17. $a_n = (-1)^n \dfrac{n}{n^3+1}$

18. $a_n = \dfrac{n^2-2\sqrt{n}+1}{1-n-3n^2}$

19. $a_n = \dfrac{e^n-e^{-n}}{e^n+e^{-n}}$

20. $a_n = n\sin\dfrac{1}{n}$

21. $a_n = \left(\dfrac{n-3}{n}\right)^n$

22. $a_n = \dfrac{n}{\ln(n+1)}$

23. $a_n = \sqrt{n+1}-\sqrt{n}$

24. $a_n = n-\sqrt{n^2-4n}$

25. $a_n = \sqrt{n^2+n}-\sqrt{n^2-1}$

26. $a_n = \left(\dfrac{n-1}{n+1}\right)^n$

27. $a_n = \dfrac{(n!)^2}{(2n)!}$

28. $a_n = \dfrac{n^2 2^n}{n!}$

29. $a_n = \dfrac{\pi^n}{1+2^{2n}}$

30. Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+2a_n}$ $(n = 1, 2, 3, \ldots)$. Show that $\{a_n\}$ is increasing and bounded above. (*Hint:* show that 3 is an upper bound.) Hence, conclude that the sequence converges, and find its limit.

∗31. Repeat Exercise 30 for the sequence defined by $a_1 = 3$, $a_{n+1} = \sqrt{15+2a_n}$, $n = 1, 2, 3, \ldots$. This time you will have to guess an upper bound.

∗32. Let $a_n = \left(1+\dfrac{1}{n}\right)^n$ so that $\ln a_n = n\ln\left(1+\dfrac{1}{n}\right)$. Use properties of the logarithm function to show that (a) $\{a_n\}$ is increasing and (b) e is an upper bound for $\{a_n\}$.

∗33. Prove Theorem 2. Also, state an analogous theorem pertaining to ultimately decreasing sequences.

34. If $\{|a_n|\}$ is bounded, prove that $\{a_n\}$ is bounded.

35. If $\lim_{n\to\infty}|a_n| = 0$, prove that $\lim_{n\to\infty} a_n = 0$.

∗36. Which of the following statements are TRUE and which are FALSE? Justify your answers.

(a) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = L > 0$, then $\lim_{n\to\infty} a_n b_n = \infty$.

(b) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty}(a_n+b_n) = 0$.

(c) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n b_n = -\infty$.

(d) If neither $\{a_n\}$ nor $\{b_n\}$ converges, then $\{a_n b_n\}$ does not converge.

(e) If $\{|a_n|\}$ converges, then $\{a_n\}$ converges.

9.2 Infinite Series

An **infinite series**, usually just called a **series**, is a formal sum of infinitely many terms; for instance,

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

is a series formed by adding the terms of the sequence $\{a_n\}$. This series is also denoted $\sum_{n=1}^{\infty} a_n$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots.$$

It is sometimes necessary or useful to start the sum from some index other than 1:

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + \cdots$$

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots.$$

Note that the latter series would make no sense if we had started the sum from $n = 1$; the first term would have been undefined.

When necessary, we can change the index of summation to start at a different value. This is accomplished by a substitution as illustrated in Example 3 of Section 5.1. For instance, using the substitution $n = m - 2$, we can rewrite $\sum_{n=1}^{\infty} a_n$ in the form $\sum_{m=3}^{\infty} a_{m-2}$. Both sums give rise to the same expansion

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots = \sum_{m=3}^{\infty} a_{m-2}.$$

Addition is an operation that is carried out on two numbers at a time. If we want to calculate the finite sum

$$a_1 + a_2 + a_3,$$

we could proceed by adding $a_1 + a_2$ and then adding a_3 to this sum, or else we might first add $a_2 + a_3$ and then add a_1 to the sum. Of course, the associative law for addition assures us we will get the same answer both ways. This is the reason the symbol $a_1 + a_2 + a_3$ makes sense; we would otherwise have to write $(a_1 + a_2) + a_3$ or $a_1 + (a_2 + a_3)$. This reasoning extends to any sum $a_1 + a_2 + \cdots + a_n$ of finitely many terms, but it is not obvious what should be meant by a sum with infinitely many terms:

$$a_1 + a_2 + a_3 + a_4 + \cdots.$$

We no longer have any assurance that the terms can be added up in any order to yield the same sum. In fact, we will see in Section 9.4 that in certain circumstances, changing the order of terms in a series can actually change the sum of the series. The interpretation we place on the infinite sum is that of adding from left to right, as suggested by the grouping

$$\cdots((((a_1 + a_2) + a_3) + a_4) + a_5) + \cdots.$$

We accomplish this by defining a new sequence $\{s_n\}$, called the **sequence of partial sums** of the series $\sum_{n=1}^{\infty} a_n$ so that s_n is the sum of the first n terms of the series:

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= s_1 + a_2 = a_1 + a_2 \\
s_3 &= s_2 + a_3 = a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= s_{n-1} + a_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{j=1}^{n} a_j. \\
&\vdots
\end{aligned}$$

We then define the sum of the infinite series to be the limit of this sequence of partial sums.

DEFINITION 3

Convergence of a series

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges to the sum** s, and we write

$$\sum_{n=1}^{\infty} a_n = s,$$

if $\lim_{n\to\infty} s_n = s$, where s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{j=1}^{n} a_j.$$

Thus, a *series* converges if and only if the *sequence* of its partial sums converges.

Similarly, a series is said to diverge to infinity, diverge to negative infinity, or simply diverge if its sequence of partial sums does so. It must be stressed that the convergence of the series $\sum_{n=1}^{\infty} a_n$ depends on the convergence of the sequence $\{s_n\} = \{\sum_{j=1}^{n} a_j\}$, *not* the sequence $\{a_n\}$.

Geometric Series

DEFINITION 4

Geometric series

A series of the form $\sum_{n=1}^{\infty} a\,r^{n-1} = a + ar + ar^2 + ar^3 + \cdots$, whose nth term is $a_n = a\,r^{n-1}$, is called a **geometric series**. The number a is the first term. The number r is called the **common ratio** of the series, since it is the value of the ratio of the $(n+1)$st term to the nth term for any $n \geq 1$:

$$\frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}} = r, \qquad n = 1, 2, 3, \ldots.$$

The nth partial sum s_n of a geometric series is calculated as follows:

$$\begin{aligned} s_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ rs_n &= \quad\;\; ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \end{aligned}$$

The second equation is obtained by multiplying the first by r. Subtracting these two equations (note the cancellations), we get $(1-r)s_n = a - ar^n$. If $r \neq 1$, we can divide by $1-r$ and get a formula for s_n.

Partial sums of geometric series

If $r = 1$, then the nth partial sum of a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is $s_n = a + a + \cdots + a = na$. If $r \neq 1$, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

If $a = 0$, then $s_n = 0$ for every n, and $\lim_{n\to\infty} s_n = 0$. Now suppose $a \neq 0$. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, so $\lim_{n\to\infty} s_n = a/(1-r)$. If $r > 1$, then $\lim_{n\to\infty} r^n = \infty$, and $\lim_{n\to\infty} s_n = \infty$ if $a > 0$, or $\lim_{n\to\infty} s_n = -\infty$ if $a < 0$. The same conclusion holds if $r = 1$, since $s_n = na$ in this case. If $r \leq -1$, $\lim_{n\to\infty} r^n$ does not exist and neither does $\lim_{n\to\infty} s_n$. Hence we conclude that

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} \text{converges to } 0 & \text{if } a = 0 \\ \text{converges to } \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges to } \infty & \text{if } r \geq 1 \text{ and } a > 0 \\ \text{diverges to } -\infty & \text{if } r \geq 1 \text{ and } a < 0 \\ \text{diverges} & \text{if } r \leq -1 \text{ and } a \neq 0. \end{cases}$$

The representation of the function $1/(1-x)$ as the sum of a geometric series,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } -1 < x < 1,$$

will be important in our discussion of power series later in this chapter.

Example 1 **(Examples of geometric series and their sums)**

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \dfrac{1}{1-\frac{1}{2}} = 2$. Here $a = 1$ and $r = \frac{1}{2}$.

Since $|r| < 1$, the series converges.

(b) $\pi - e + \frac{e^2}{\pi} - \frac{e^3}{\pi^2} + \cdots = \sum_{n=1}^{\infty} \pi \left(-\frac{e}{\pi}\right)^{n-1}$ Here $a = \pi$ and $r = -\frac{e}{\pi}$.

$$= \frac{\pi}{1-\left(-\frac{e}{\pi}\right)} = \frac{\pi^2}{\pi + e}.$$

The series converges since $\left|-\frac{e}{\pi}\right| < 1$.

(c) $1 + 2^{1/2} + 2 + 2^{3/2} + \cdots = \sum_{n=1}^{\infty} (\sqrt{2})^{n-1}$. This series diverges to ∞ since $a = 1 > 0$ and $r = \sqrt{2} > 1$.

(d) $1 - 1 + 1 - 1 + 1 - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}$. This series diverges since $r = -1$.

(e) Let $x = 0.323232\cdots = 0.\overline{32}$; then

$$x = \frac{32}{100} + \frac{32}{100^2} + \frac{32}{100^3} + \cdots = \sum_{n=1}^{\infty} \frac{32}{100}\left(\frac{1}{100}\right)^{n-1} = \frac{32}{100}\,\frac{1}{1-\frac{1}{100}} = \frac{32}{99}.$$

This is an alternative to the method of Example 1 of Section P.1 for representing repeating decimals as quotients of integers. ■

Example 2 If money earns interest at a constant effective rate of 8% per year, how much should you pay today for an annuity that will pay you (a) \$1,000 at the end of each of the next 10 years and (b) \$1,000 at the end of every year forever?

Solution A payment of \$1,000 that is due to be received n years from now has present value $\$1{,}000 \times \left(\frac{1}{1.08}\right)^n$ (since $\$A$ would grow to $\$A(1.08)^n$ in n years). Thus \$1,000 payments at the end of each of the next n years are worth $\$s_n$ at the present time, where

$$\begin{aligned} s_n &= 1{,}000\left[\frac{1}{1.08} + \left(\frac{1}{1.08}\right)^2 + \cdots + \left(\frac{1}{1.08}\right)^n\right] \\ &= \frac{1{,}000}{1.08}\left[1 + \frac{1}{1.08} + \left(\frac{1}{1.08}\right)^2 + \cdots + \left(\frac{1}{1.08}\right)^{n-1}\right]. \\ &= \frac{1{,}000}{1.08}\,\frac{1-\left(\frac{1}{1.08}\right)^n}{1-\frac{1}{1.08}} = \frac{1{,}000}{0.08}\left[1-\left(\frac{1}{1.08}\right)^n\right] \end{aligned}$$

(a) The present value of 10 future payments is $\$s_{10} = \$6{,}710.08$.

(b) The present value of future payments continuing forever is

$$\$\lim_{n\to\infty} s_n = \frac{\$1{,}000}{0.08} = \$12{,}500.$$

Telescoping Series and Harmonic Series

Example 3 Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \cdots$$

converges and find its sum.

Solution Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we can write the partial sum s_n in the form

$$\begin{aligned} s_n &= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \\ &= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots \\ &\quad \cdots + \left(\frac{1}{n-1}-\frac{1}{n}\right) + \left(\frac{1}{n}-\frac{1}{n+1}\right) \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} s_n = 1$ and the series converges to 1:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

This is an example of a **telescoping series**, so called because the partial sums *fold up* into a simple form when the terms are expanded in partial fractions. Other examples can be found in the exercises at the end of this section. As these examples show, the method of partial fractions can be a useful tool for series as well as for integrals.

Example 4 Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges to infinity.

Solution If s_n is the nth partial sum of the harmonic series, then

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &= \text{sum of areas of rectangles shaded in Figure 9.3} \\ &> \text{area under } y = \frac{1}{x} \text{ from 1 to } n+1 \\ &= \int_1^{n+1} \frac{dx}{x} = \ln(n+1). \end{aligned}$$

Now $\lim_{n\to\infty} \ln(n+1) = \infty$. Therefore, $\lim_{n\to\infty} s_n = \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \qquad \text{diverges to infinity.}$$

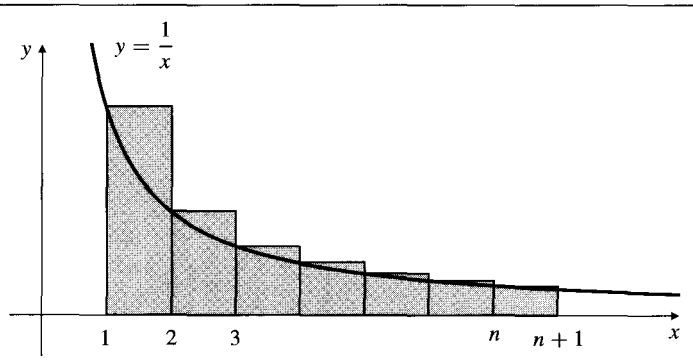

Figure 9.3 A partial sum of the harmonic series

Like geometric series, the harmonic series will often be encountered in subsequent sections.

Some Theorems About Series

THEOREM 4 If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

PROOF If $s_n = a_1 + a_2 + \cdots + a_n$, then $s_n - s_{n-1} = a_n$. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} s_n = s$ exists, and $\lim_{n\to\infty} s_{n-1} = s$. Hence $\lim_{n\to\infty} a_n = s - s = 0$.

Remark Theorem 4 is *very important* for the understanding of infinite series. Students often err either in forgetting that *a series cannot converge if its terms do not approach zero* or in confusing this result with its *converse*, which is false. The converse would say that if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ must converge. The harmonic series is a counterexample showing the falsehood of this assertion:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{but} \qquad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges to infinity.}$$

When considering whether a given series converges, the first question you should ask yourself is: "Does the nth term approach 0 as n approaches ∞?" If the answer is *no*, then the series does *not* converge. If the answer is *yes*, then the series *may or may not* converge. If the sequence of terms $\{a_n\}$ tends to a nonzero limit L, then $\sum_{n=1}^{\infty} a_n$ diverges to infinity if $L > 0$ and diverges to negative infinity if $L < 0$.

Example 5

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n-1}$ diverges to infinity since $\lim_{n\to\infty} \dfrac{n}{2n-1} = 1/2 > 0$.

(b) $\sum_{n=1}^{\infty} (-1)^n n \sin(1/n)$ diverges since

$$\lim_{n\to\infty} \left| (-1)^n n \sin \frac{1}{n} \right| = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0+} \frac{\sin x}{x} = 1 \neq 0.$$

The following theorem asserts that it is only the *ultimate* behaviour of $\{a_n\}$ that determines whether $\sum_{n=1}^{\infty} a_n$ converges. Any finite number of terms can be dropped from the beginning of a series without affecting the convergence; the convergence depends only on the *tail* of the series. Of course, the actual sum of the series depends on *all* the terms.

THEOREM 5 $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges for any integer $N \geq 1$.

THEOREM 6 If $\{a_n\}$ is ultimately positive, then the series $\sum_{n=1}^{\infty} a_n$ must either converge (if its partial sums are bounded above) or diverge to infinity (if its partial sums are not bounded above).

The proofs of these two theorems are posed as exercises at the end of this section. The following theorem is just a reformulation of standard laws of limits.

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to A and B, respectively, then

(a) $\sum_{n=1}^{\infty} ca_n$ converges to cA (where c is any constant);

(b) $\sum_{n=1}^{\infty} (a_n \pm b_n)$ converges to $A \pm B$;

(c) if $a_n \leq b_n$ for all $n = 1, 2, 3, \ldots$, then $A \leq B$.

Example 6 Find the sum of the series $\sum_{n=1}^{\infty} \frac{1+2^{n+1}}{3^n}$.

Solution The given series is the sum of two geometric series,

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1/3}{1-(1/3)} = \frac{1}{2} \quad \text{and}$$

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{4}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{4/3}{1-(2/3)} = 4.$$

Thus its sum is $\frac{1}{2} + 4 = \frac{9}{2}$ by Theorem 7(b).

Exercises 9.2

In Exercises 1–18, find the sum of the given series, or show that the series diverges (possibly to infinity or negative infinity). Exercises 11–14 are telescoping series and should be done by partial fractions as suggested in Example 3 in this section.

1. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{n=1}^{\infty} \frac{1}{3^n}$

2. $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \cdots = \sum_{n=1}^{\infty} 3\left(-\frac{1}{4}\right)^{n-1}$

3. $\sum_{n=5}^{\infty} \frac{1}{(2+\pi)^{2n}}$

4. $\sum_{n=0}^{\infty} \frac{5}{10^{3n}}$

5. $\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{2n}}$

6. $\sum_{n=0}^{\infty} \frac{1}{e^n}$

7. $\sum_{k=0}^{\infty} \frac{2^{k+3}}{e^{k-3}}$

8. $\sum_{j=1}^{\infty} \pi^{j/2} \cos(j\pi)$

9. $\sum_{n=1}^{\infty} \frac{3+2^n}{2^{n+2}}$

10. $\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}}$

11. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots$

12. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots$

13. $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} = \frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \cdots$

*** 14.** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$
$= \frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \cdots$

15. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

16. $\sum_{n=1}^{\infty} \frac{n}{n+2}$

17. $\sum_{n=1}^{\infty} n^{-1/2}$

18. $\sum_{n=1}^{\infty} \frac{2}{n+1}$

19. Obtain a simple expression for the partial sum s_n of the series $\sum_{n=1}^{\infty}(-1)^n$, and use it to show that the series diverges.

20. Find the sum of the series

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots.$$

21. When dropped, an elastic ball bounces back up to a height three-quarters of that from which it fell. If the ball is dropped from a height of 2 m and allowed to bounce up and down indefinitely, what is the total distance it travels before coming to rest?

22. If a bank account pays 10% simple interest into an account once a year, what is the balance in the account at the end of 8 years if \$1,000 is deposited into the account at the beginning of each of the 8 years? (Assume there was no balance in the account initially.)

*** 23.** Prove Theorem 5.

*** 24.** Prove Theorem 6.

*** 25.** State a theorem analogous to Theorem 6 but for a negative sequence.

In Exercises 26–31, decide whether the given statement is TRUE or FALSE. If it is true, prove it. If it is false, give a counterexample showing the falsehood.

*** 26.** If $a_n = 0$ for every n, then $\sum a_n$ converges.

*** 27.** If $\sum a_n$ converges, then $\sum(1/a_n)$ diverges to infinity.

*** 28.** If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum(a_n + b_n)$.

*** 29.** If $a_n \geq c > 0$ for every n, then $\sum a_n$ diverges to infinity.

* **30.** If $\sum a_n$ diverges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ diverges.

* **31.** If $a_n > 0$ and $\sum a_n$ converges, then $\sum (a_n)^2$ converges.

9.3 Convergence Tests for Positive Series

In the previous section we saw a few examples of convergent series (geometric and telescoping series) whose sums could be determined exactly because the partial sums s_n could be expressed in closed form as explicit functions of n whose limits as $n \to \infty$ could be evaluated. It is not usually possible to do this with a given series, and therefore it is not usually possible to determine the sum of the series exactly. However, there are many techniques for determining whether a given series converges and, if it does, for approximating the sum to any desired degree of accuracy.

In this section we deal exclusively with *positive series*, that is, series of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots,$$

where $a_n \geq 0$ for all $n \geq 1$. As noted in Theorem 6, such a series will converge if its partial sums are bounded above and will diverge to infinity otherwise. All our results apply equally well to *ultimately* positive series since convergence or divergence depends only on the *tail* of a series.

The Integral Test

The integral test provides a means for determining whether an ultimately positive series converges or diverges by comparing it with an improper integral that behaves similarly. Example 4 in Section 9.2 is an example of the use of this technique. We formalize the method in the following theorem.

THEOREM 8

The integral test

Suppose that $a_n = f(n)$, where f is positive, continuous, and nonincreasing on an interval $[N, \infty[$ for some positive integer N. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(t)\,dt$$

either both converge or both diverge to infinity.

PROOF Let $s_n = a_1 + a_2 + \cdots + a_n$. If $n > N$, we have

$$\begin{aligned} s_n &= s_N + a_{N+1} + a_{N+2} + \cdots + a_n \\ &= s_N + f(N+1) + f(N+2) + \cdots + f(n) \\ &= s_N + \text{ sum of areas of rectangles shaded in Figure 9.4(a)} \\ &\leq s_N + \int_N^{\infty} f(t)\,dt. \end{aligned}$$

If the improper integral $\int_N^{\infty} f(t)\,dt$ converges, then the sequence $\{s_n\}$ is bounded above and $\sum_{n=1}^{\infty} a_n$ converges.

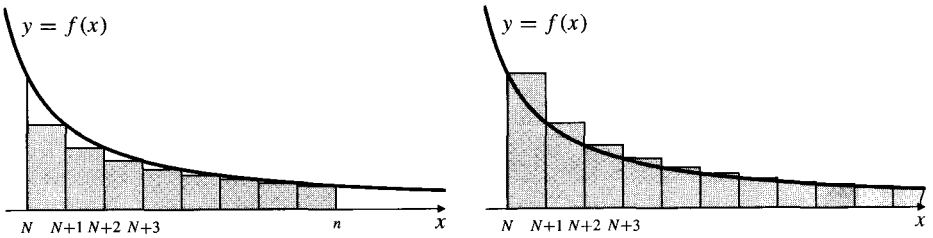

Figure 9.4 (a) (b)

Conversely, suppose that $\sum_{n=1}^{\infty} a_n$ converges to the sum s. Then

$$\begin{aligned}\int_N^{\infty} f(t)\,dt &= \text{area under } y = f(t) \text{ above } y = 0 \text{ from } t = N \text{ to } t = \infty \\ &\le \text{sum of areas of shaded rectangles in Figure 9.4(b)} \\ &= a_N + a_{N+1} + a_{N+2} + \cdots \\ &= s - s_{N-1} < \infty,\end{aligned}$$

so the improper integral represents a finite area and is thus convergent. (We omit the remaining details showing that $\lim_{R\to\infty} \int_N^R f(t)\,dt$ exists; like the series case, the argument depends on the completeness of the real numbers.)

Remark If $a_n = f(n)$, where f is positive, continuous, and nonincreasing on $[1, \infty[$, then Theorem 8 assures us that $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge to infinity. It does *not* tell us that the sum of the series is equal to the value of the integral. The two are not likely to be equal in the case of convergence. However, as we see below, integrals can help us approximate the sum of a series.

The principal use of the integral test is to establish the result of the following example concerning the series $\sum_{n=1}^{\infty} n^{-p}$, which is called a ***p*-series**. This result should be memorized; we will frequently compare the behaviour of other series with p-series later in this and subsequent sections.

Example 1 (***p*-series**) Show that

$$\sum_{n=1}^{\infty} n^{-p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges to infinity if } p \le 1. \end{cases}$$

Solution Observe that if $p > 0$, then $f(x) = x^{-p}$ is positive, continuous, and decreasing on $[1, \infty[$. By the integral test, the p-series converges for $p > 1$ and diverges for $0 < p \le 1$ by comparison with $\int_1^{\infty} x^{-p}\,dx$. (See Theorem 2(a) of Section 6.5.) If $p \le 0$, then $\lim_{n\to\infty}(1/n^p) \ne 0$, so the series cannot converge in this case. Being a positive series, it must diverge to infinity. ■

Remark The harmonic series $\sum_{n=1}^{\infty} n^{-1}$ (the case $p = 1$ of the p-series) is on the borderline between convergence and divergence, although it diverges. While its terms decrease toward 0 as n increases, they do not decrease *fast enough* to allow the sum of the series to be finite. If $p > 1$, the terms of $\sum_{n=1}^{\infty} n^{-p}$ decrease toward

zero fast enough that their sum is finite. We can refine the distinction between convergence and divergence at $p = 1$ by using terms that decrease faster than $1/n$, but not as fast as $1/n^q$ for any $q > 1$. If $p > 0$, terms $1/\big(n(\ln n)^p\big)$ have this property since $\ln n$ grows more slowly than any positive power of n as n increases. The question now arises whether $\sum_{n=2}^{\infty} 1/(n(\ln n)^p)$ converges or not. It does, provided again that $p > 1$; you can use the substitution $u = \ln x$ to check that

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} \frac{du}{u^p},$$

which converges if $p > 1$ and diverges if $0 < p \leq 1$. This process of fine-tuning Example 1 can be extended even further. (See Exercise 36 below.)

Using Integral Bounds to Estimate the Sum of a Series

Suppose that $a_k = f(k)$ for $k = n+1, n+2, n+3, \ldots$, where f is a positive, continuous function, decreasing at least on the interval $[n, \infty[$. We have:

$$\begin{aligned} s - s_n &= \sum_{k=n+1}^{\infty} f(k) \\ &= \text{sum of areas of rectangles shaded in Figure 9.5(a)} \\ &\leq \int_n^{\infty} f(x)\,dx. \end{aligned}$$

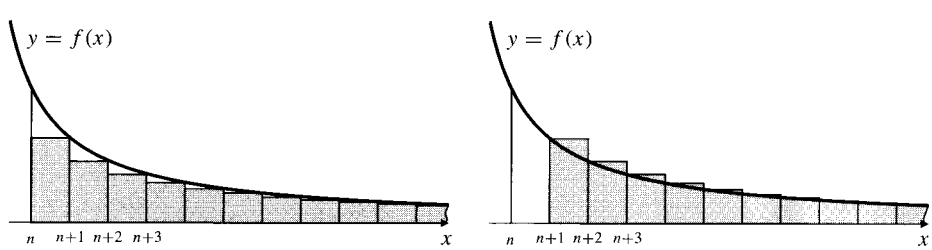

Figure 9.5 (a) (b)

Similarly,

$$\begin{aligned} s - s_n &= \text{sum of areas of rectangles in Figure 9.5(b)} \\ &\geq \int_{n+1}^{\infty} f(x)\,dx. \end{aligned}$$

If we define

$$A_n = \int_n^{\infty} f(x)\,dx,$$

then we can combine the above inequalities to obtain

$$A_{n+1} \leq s - s_n \leq A_n,$$

or, equivalently:

$$s_n + A_{n+1} \leq s \leq s_n + A_n.$$

The error in the approximation $s \approx s_n$ satisfies $0 \le s - s_n \le A_n$. However, since s must lie in the interval $[s_n + A_{n+1}, s_n + A_n]$, we can do better by using the midpoint s_n^* of this interval as an approximation for s. The error is then less than half the length $A_n - A_{n+1}$ of the interval:

A better integral approximation

The error $|s - s_n^*|$ in the approximation

$$s \approx s_n^* = s_n + \frac{A_{n+1} + A_n}{2}, \qquad \text{where} \quad A_n = \int_n^\infty f(x)\,dx,$$

satisfies $\quad |s - s_n^*| \le \dfrac{A_n - A_{n+1}}{2}.$

(Whenever a quantity is known to lie in a certain interval, the midpoint of that interval can be used to approximate the quantity, and the absolute value of the error in that approximation does not exceed half the length of the interval.)

Example 2 Find the best approximation s_n^* to the sum s of the series $\sum_{n=1}^\infty 1/n^2$, making use of the partial sum s_n of the first n terms. How large would n have to be to ensure that the approximation $s \approx s_n^*$ has error less than 0.001 in absolute value? How large would n have to be to ensure that the approximation $s \approx s_n$ has error less than 0.001 in absolute value?

Solution Since $f(x) = 1/x^2$ is positive, continuous, and decreasing on $[1, \infty[$ for any $n = 1, 2, 3, \ldots$, we have

$$s_n + A_{n+1} \le s \le s_n + A_n,$$

where

$$A_n = \int_n^\infty \frac{dx}{x^2} = \lim_{R\to\infty} \left(-\frac{1}{x}\right)\Bigg|_n^R = \frac{1}{n}.$$

The best approximation to s using s_n is

$$\begin{aligned} s_n^* &= s_n + \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right) = s_n + \frac{2n+1}{2n(n+1)} \\ &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{2n+1}{2n(n+1)}. \end{aligned}$$

The error in this approximation satisfies

$$|s - s_n^*| \le \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2n(n+1)} \le 0.001,$$

provided $2n(n+1) \ge 1/0.001 = 1{,}000$. It is easily checked that this condition is satisfied if $n \ge 22$; the approximation

$$s \approx s_{22}^* = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{22^2} + \frac{45}{44 \times 23}$$

will have error with absolute value not exceeding 0.001. Had we used the approximation $s \approx s_n$ we could only have concluded that

$$0 \le s - s_n \le A_n = \frac{1}{n} < 0.001,$$

provided $n > 1{,}000$; we would need 1,000 terms of the series to get the desired accuracy.

Comparison Tests

The next test we consider for positive series is analogous to the comparison theorem for improper integrals. (See Theorem 3 of Section 6.5.) It enables us to determine the convergence or divergence of one series by comparing it with another series that is known to converge or diverge.

THEOREM 9

A comparison test

Let $\{a_n\}$ and $\{b_n\}$ be sequences for which there exists a positive constant K such that, ultimately, $0 \le a_n \le Kb_n$.

(a) If the series $\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

(b) If the series $\sum_{n=1}^{\infty} a_n$ diverges to infinity, then so does the series $\sum_{n=1}^{\infty} b_n$.

BEWARE

Theorem 9 does *not* say that if $\sum a_n$ converges, then $\sum b_n$ converges. It is possible that the *smaller* sum may be finite while the *larger* one is infinite. (Do not confuse a theorem with its converse.)

PROOF Since a series converges if and only if its tail converges (Theorem 5), we can assume, without loss of generality, that the condition $0 \le a_n \le Kb_n$ holds for all $n \ge 1$. Let $s_n = a_1 + a_2 + \cdots + a_n$ and $S_n = b_1 + b_2 + \cdots + b_n$. Then $s_n \le KS_n$. If $\sum b_n$ converges, then $\{S_n\}$ is convergent and hence is bounded by Theorem 1. Hence $\{s_n\}$ is bounded above. By Theorem 6, $\sum a_n$ converges. Since the convergence of $\sum b_n$ guarantees that of $\sum a_n$, if the latter series diverges to infinity, then the former cannot converge either, so it must diverge to infinity too.

Example 3 Do the following series converge or not? Give reasons for your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$, (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$.

Solution In each case we must find a suitable comparison series that we already know converges or diverges.

(a) Since $0 < \dfrac{1}{2^n+1} < \dfrac{1}{2^n}$ for $n = 1, 2, 3, \ldots$, and since $\sum_{n=1}^{\infty} \dfrac{1}{2^n}$ is a convergent geometric series, the series $\sum_{n=1}^{\infty} \dfrac{1}{2^n+1}$ also converges by comparison.

(b) Observe that $\dfrac{3n+1}{n^3+1}$ behaves like $\dfrac{3}{n^2}$ for large n, so we would expect to compare the series with the convergent p-series $\sum_{n=1}^{\infty} n^{-2}$. We have, for $n \ge 1$,

$$\frac{3n+1}{n^3+1} = \frac{3n}{n^3+1} + \frac{1}{n^3+1} < \frac{3n}{n^3} + \frac{1}{n^3} < \frac{3}{n^2} + \frac{1}{n^2} = \frac{4}{n^2}.$$

Thus, the given series converges by Theorem 9.

(c) For $n = 2, 3, 4, \ldots$, we have $0 < \ln n < n$. Thus $\dfrac{1}{\ln n} > \dfrac{1}{n}$. Since $\sum_{n=2}^{\infty} \dfrac{1}{n}$ diverges to infinity (it is a harmonic series), so does $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ by comparison. ■

The following theorem provides a version of the comparison test that is not quite as general as Theorem 9 but is often easier to apply in specific cases.

THEOREM 10 **A limit comparison test**

Suppose that $\{a_n\}$ and $\{b_n\}$ are positive sequences and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where L is either a nonnegative finite number or $+\infty$.

(a) If $L < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(b) If $L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges to infinity, then so does $\sum_{n=1}^{\infty} a_n$.

PROOF If $L < \infty$, then for n sufficiently large, we have $b_n > 0$ and

$$0 \le \frac{a_n}{b_n} \le L + 1,$$

so $0 \le a_n \le (L+1)b_n$. Hence $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges, by Theorem 9(a).

If $L > 0$, then for n sufficiently large

$$\frac{a_n}{b_n} \ge \frac{L}{2}.$$

Therefore, $0 < b_n \le (2/L)a_n$, and $\sum_{n=1}^{\infty} a_n$ diverges to infinity if $\sum_{n=1}^{\infty} b_n$ does, by Theorem 9(b).

Example 4 Do the following series converge or not? Give reasons for your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3 - 2n + 3}$.

Solution Again we must make appropriate choices for comparison series.

(a) The terms of this series decrease like $1/\sqrt{n}$. Observe that

$$L = \lim_{n\to\infty} \frac{\dfrac{1}{1+\sqrt{n}}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{(1/\sqrt{n})+1} = 1.$$

Since the p-series $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges to infinity ($p = 1/2$), so does the series $\sum_{n=1}^{\infty} \dfrac{1}{1+\sqrt{n}}$, by the limit comparison test.

(b) For large n, the terms behave like n/n^3, so let us compare the series with the p-series $\sum_{n=1}^{\infty} 1/n^2$, which we know converges.

$$L = \lim_{n\to\infty} \frac{\dfrac{n+5}{n^3-2n+3}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+5n^2}{n^3-2n+3} = 1.$$

Since $L < \infty$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ also converges by the limit comparison test. ■

In order to apply the original version of the comparison test (Theorem 9) successfully, it is important to have an intuitive feeling for whether the given series converges or diverges. The form of the comparison will depend on whether you are trying to prove convergence or divergence. For instance, if you did not know intuitively that

$$\sum_{n=1}^{\infty} \frac{1}{100n+20{,}000}$$

would have to diverge to infinity, you might try to argue that

$$\frac{1}{100n+20{,}000} < \frac{1}{n} \qquad \text{for } n = 1, 2, 3, \ldots.$$

While true, this doesn't help at all. $\sum_{n=1}^{\infty} 1/n$ diverges to infinity; therefore Theorem 9 yields no information from this comparison. We could, of course, argue instead that

$$\frac{1}{100n+20{,}000} \geq \frac{1}{20{,}100n} \qquad \text{if } n \geq 1,$$

and conclude by Theorem 9 that $\sum_{n=1}^{\infty}(1/(100n+20{,}000))$ diverges to infinity by comparison with the divergent series $\sum_{n=1}^{\infty} 1/n$. An easier way is to use Theorem 10 and the fact that

$$L = \lim_{n\to\infty} \frac{\dfrac{1}{100n+20{,}000}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{100n+20{,}000} = \frac{1}{100} > 0.$$

However, the limit comparison test Theorem 10 has a disadvantage when compared to the ordinary comparison test Theorem 9. It can fail in certain cases because the limit L does not exist. In such cases it is possible that the ordinary comparison test may still work.

Example 5 Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$ for convergence.

Solution Since

$$\lim_{n\to\infty} \frac{\dfrac{1+\sin n}{n^2}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} (1+\sin n)$$

does not exist, the limit comparison test gives us no information. However, since $\sin n \le 1$, we have

$$0 \le \frac{1+\sin n}{n^2} \le \frac{2}{n^2} \qquad \text{for } n = 1, 2, 3, \ldots.$$

The given series does, in fact, converge by comparison with $\sum_{n=1}^{\infty} 1/n^2$, using the ordinary comparison test.

The Ratio and Root Tests

THEOREM 11

The ratio test

Suppose that $a_n > 0$ (ultimately) and that $\rho = \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists or is $+\infty$.

(a) If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $1 < \rho \le \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(c) If $\rho = 1$, this test gives no information; the series may either converge or diverge to infinity.

PROOF Here ρ is the lowercase Greek letter *rho*, (pronounced "roe").

(a) Suppose $\rho < 1$. Pick a number r such that $\rho < r < 1$. Since we are given that $\lim_{n\to\infty} a_{n+1}/a_n = \rho$, we have $a_{n+1}/a_n \le r$ for n sufficiently large; that is, $a_{n+1} \le r a_n$ for $n \ge N$, say. In particular,

$$\begin{aligned}
a_{N+1} &\le r a_N \\
a_{N+2} &\le r a_{N+1} \le r^2 a_N \\
a_{N+3} &\le r a_{N+2} \le r^3 a_N \\
&\vdots \\
a_{N+k} &\le r^k a_N \qquad (k = 0, 1, 2, 3, \ldots)
\end{aligned}$$

Hence, $\sum_{n=N}^{\infty} a_n$ converges by comparison with the convergent geometric series $\sum_{k=0}^{\infty} r^k$. It follows that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$ must also converge.

(b) Now suppose that $\rho > 1$. Pick a number r such that $1 < r < \rho$. Since $\lim_{n\to\infty} a_{n+1}/a_n = \rho$, we have $a_{n+1}/a_n \ge r$ for n sufficiently large, say for $n \ge N$. We assume N is chosen large enough that $a_N > 0$. It follows by an argument similar to that used in part (a) that $a_{N+k} \ge r^k a_N$ for $k = 0, 1, 2, \ldots$, and since $r > 1$, $\lim_{n\to\infty} a_n = \infty$. Therefore $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(c) If ρ is computed for the series $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} 1/n^2$, we get $\rho = 1$ in each case. Since the first series diverges to infinity and the second converges, the ratio test cannot distinguish between convergence and divergence if $\rho = 1$.

All p-series fall into the indecisive category where $\rho = 1$, as does $\sum_{n=1}^{\infty} a_n$, where a_n is any rational function of n. The ratio test is most useful for series whose terms decrease at least exponentially fast. The presence of factorials in a term also suggests that the ratio test might be useful.

Example 6 Test the following series for convergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{99^n}{n!}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{2^n}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$.

Solution We use the ratio test for each of these series.

(a) $$\rho = \lim_{n\to\infty} \frac{99^{n+1}}{(n+1)!} \bigg/ \frac{99^n}{n!} = \lim_{n\to\infty} \frac{99}{n+1} = 0 < 1.$$

Thus $\sum_{n=1}^{\infty}(99^n/n!)$ converges.

(b) $$\rho = \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \bigg/ \frac{n^5}{2^n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^5 = \frac{1}{2} < 1.$$

Hence $\sum_{n=1}^{\infty}(n^5/2^n)$ converges.

(c) $$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \bigg/ \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$
$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

Thus $\sum_{n=1}^{\infty}(n!/n^n)$ converges.

(d) $$\rho = \lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} \bigg/ \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4 > 1.$$

Thus $\sum_{n=1}^{\infty}(2n)!/(n!)^2$ diverges to infinity. ■

The following theorem is very similar to the ratio test but is less frequently used. Its proof is left as an exercise. (See Exercise 37.) For examples of series to which it can be applied, see Exercises 38 and 39.

THEOREM 12 **The root test**

Suppose that $a_n > 0$ (ultimately) and that $\sigma = \lim_{n\to\infty}(a_n)^{1/n}$ exists or is $+\infty$.

(a) If $0 \le \sigma < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $1 < \sigma \le \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(c) If $\sigma = 1$, this test gives no information; the series may either converge or diverge to infinity.

Using Geometric Bounds to Estimate the Sum of a Series

Suppose that an inequality of the form

$$0 \le a_k \le Kr^k$$

holds for $k = n+1, n+2, n+3, \ldots$, where K and r are constants and $r < 1$. We can then use a geometric series to bound the tail of $\sum_{n=1}^{\infty} a_n$.

$$\begin{aligned} 0 \le s - s_n &= \sum_{k=n+1}^{\infty} a_k \le \sum_{k=n+1}^{\infty} Kr^k \\ &= Kr^{n+1}(1 + r + r^2 + \cdots) \\ &= \frac{Kr^{n+1}}{1-r}. \end{aligned}$$

Since $r < 1$, the series converges and the error approaches 0 at an exponential rate as n increases.

Example 7 In Section 9.6 we will show that

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

(Recall that $0! = 1$.) Estimate the error if the sum s_n of the first n terms of the series is used to approximate e. Find e to 3-decimal-place accuracy using the series.

Solution We have

$$\begin{aligned} s_n &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{(n-1)!}. \end{aligned}$$

(Since the series starts with the term for $n = 0$, the nth term is $1/(n-1)!$.) We can estimate the error in the approximation $s \approx s_n$ as follows:

$$\begin{aligned} 0 < s - s_n &= \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &= \frac{1}{n!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right) \\ &< \frac{1}{n!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right) \end{aligned}$$

since $n+2 > n+1$, $n+3 > n+1$, and so on. The latter series is geometric, so

$$0 < s - s_n < \frac{1}{n!}\,\frac{1}{1 - \dfrac{1}{n+1}} = \frac{n+1}{n!n}.$$

If we want to evaluate e accurately to 3 decimal places, then we must ensure that the error is less than 5 in the fourth decimal place, that is, that the error is less than 0.0005. Hence we want

$$\frac{n+1}{n}\,\frac{1}{n!} < 0.0005 = \frac{1}{2{,}000}.$$

Since $7! = 5{,}040$, but $6! = 720$, we can use $n = 7$ but no smaller. We have

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$$
$$= 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \approx 2.718 \quad \text{to 3 decimal places.}$$

■

It is appropriate to use geometric series to bound the tails of positive series whose convergence would be demonstrated by the ratio test. Such series converge ultimately faster than any p-series $\sum_{n=1}^{\infty} n^{-p}$, for which the limit ratio is $\rho = 1$.

Exercises 9.3

In Exercises 1–26, determine whether the given series converges or diverges by using any appropriate test. The p-series can be used for comparison, as can geometric series. Be alert for series whose terms do not approach 0.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4-2}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$

5. $\displaystyle\sum_{n=1}^{\infty} \left|\sin \frac{1}{n^2}\right|$

6. $\displaystyle\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(3n)}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{1+n}{2+n}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

13. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \sqrt{\ln \ln n}}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n (\ln \ln n)^2}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^4}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n(n+1)}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2 e^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{n^{100}2^n}{\sqrt{n!}}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

25. $\displaystyle\sum_{n=4}^{\infty} \frac{2^n}{3^n - n^3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$

In Exercises 27–30, use s_n and integral bounds to find the smallest interval that you can be sure contains the sum s of the series. If the midpoint s_n^* of this interval is used to approximate s, how large should n be chosen to ensure that the error is less than 0.001?

27. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$

28. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$

29. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

30. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+4}$

For each positive series in Exercises 31–34, find the best upper bound you can for the error $s - s_n$ encountered if the partial sum s_n is used to approximate the sum s of the series. How many terms of each series do you need to be sure that the approximation has error less than 0.001?

31. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k k!}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)!}$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

35. Use the integral test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Show that the sum s of the series is less than $\pi/2$.

* 36. Show that $\sum_{n=3}^{\infty}(1/(n \ln n (\ln \ln n)^p)$ converges if and only if $p > 1$. Generalize this result to series of the form

$$\sum_{n=N}^{\infty} \frac{1}{n(\ln n)(\ln \ln n) \cdots (\ln_j n)(\ln_{j+1} n)^p},$$

where $\ln_j n = \underbrace{\ln\ln\ln\ln\cdots\ln}_{j\ \ln's} n$.

* **37.** Prove the root test. *Hint:* mimic the proof of the ratio test.

38. Use the root test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges.

* **39.** Use the root test to test the following series for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}.$$

40. Repeat Exercise 38, but use the ratio test instead of the root test.

* **41.** Try to use the ratio test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happens? Now observe that

$$\frac{2^{2n}(n!)^2}{(2n)!} = \frac{[2n(2n-2)(2n-4)\cdots 6\times 4\times 2]^2}{2n(2n-1)(2n-2)\cdots 4\times 3\times 2\times 1}$$
$$= \frac{2n}{2n-1}\times\frac{2n-2}{2n-3}\times\cdots\times\frac{4}{3}\times\frac{2}{1}.$$

Does the given series converge? Why or why not?

* **42.** Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$ converges. *Hint:* proceed as in Exercise 41. Show that $a_n \geq 1/(2n)$.

* **43.** (a) Show that if $k > 0$ and n is a positive integer, then $n < \frac{1}{k}(1+k)^n$.

(b) Use the estimate in (a) with $0 < k < 1$ to obtain an upper bound for the sum of the series $\sum_{n=0}^{\infty} n/2^n$. For what value of k is this bound least?

(c) If we use the sum s_n of the first n terms to approximate the sum s of the series in (b), obtain an upper bound for the error $s - s_n$ using the inequality from (a). For given n, find k to minimize this upper bound.

* **44.** **(Improving the convergence of a series)** We know that $\sum_{n=1}^{\infty} 1/(n(n+1)) = 1$. (See Example 3 of Section 9.2.) Since

$$\frac{1}{n^2} = \frac{1}{n(n+1)} + c_n, \quad \text{where} \quad c_n = \frac{1}{n^2(n+1)},$$

we have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} c_n$.

The series $\sum_{n=1}^{\infty} c_n$ converges more rapidly than does $\sum_{n=1}^{\infty} 1/n^2$ because its terms decrease like $1/n^3$. Hence, fewer terms of that series will be needed to compute $\sum_{n=1}^{\infty} 1/n^2$ to any desired degree of accuracy than would be needed if we calculated with $\sum_{n=1}^{\infty} 1/n^2$ directly. Using integral upper and lower bounds, determine a value of n for which the modified partial sum s_n^* for the series $\sum_{n=1}^{\infty} c_n$ approximates the sum of that series with error less than 0.001 in absolute value. Hence, determine $\sum_{n=1}^{\infty} 1/n^2$ to within 0.001 of its true value.

(The technique exibited in this exercise is known as *improving the convergence* of a series. It can be applied to estimating the sum $\sum a_n$ if we know the sum $\sum b_n$ and if $a_n - b_n = c_n$, where $|c_n|$ decreases faster than $|a_n|$ as n tends to infinity.)

9.4 Absolute and Conditional Convergence

All of the series $\sum_{n=1}^{\infty} a_n$ considered in the previous section were ultimately positive; that is, $a_n \geq 0$ for n sufficiently large. We now drop this restriction and allow arbitrary real terms a_n. We can, however, always obtain a positive series from any given series by replacing all the terms with their absolute values.

DEFINITION 5

Absolute convergence

The series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

The series

$$s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots$$

converges absolutely since

$$S = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

converges. It seems reasonable that the first series must converge, and its sum s should satisfy $-S \le s \le S$. In general, the cancellation that occurs because some terms are negative and others positive makes it *easier* for a series to converge than if all the terms are of one sign. We verify this insight in the following theorem.

THEOREM 13 If a series converges absolutely, then it converges.

PROOF Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent, and let $b_n = a_n + |a_n|$ for each n. Since $-|a_n| \le a_n \le |a_n|$, we have $0 \le b_n \le 2|a_n|$ for each n. Thus $\sum_{n=1}^{\infty} b_n$ converges by the comparison test. Therefore, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |a_n|$ also converges.

Again you are cautioned not to confuse the statement of Theorem 13 with the converse statement, which is false. We will show later in this section that the **alternating harmonic series**

BEWARE

Although absolute convergence implies convergence, convergence does *not* imply absolute convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges, although it does not converge absolutely. If we replace all the terms by their absolute values, we get the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty.$$

DEFINITION 6

Conditional convergence

If $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent, then we say that it is **conditionally convergent** or that it **converges conditionally**.

The alternating harmonic series is an example of a conditionally convergent series.

The comparison tests, the integral test, and the ratio test, can each be used to test for absolute convergence. They should be applied to the series $\sum_{n=1}^{\infty} |a_n|$. For the ratio test we calculate $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$. If $\rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, then $\lim_{n\to\infty} |a_n| = \infty$, so both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ must diverge. If $\rho = 1$, we get no information; the series $\sum_{n=1}^{\infty} a_n$ may converge absolutely, it may converge conditionally, or it may diverge.

Example 1 Test the following series for absolute convergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$.

Solution

(a) $\displaystyle\lim_{n\to\infty} \left|\frac{(-1)^{n-1}}{2n-1}\right| \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} > 0$. Since the harmonic series $\sum_{n=1}^{\infty}(1/n)$ diverges to infinity, the comparison test assures us that $\sum_{n=1}^{\infty}((-1)^{n-1}/(2n-1))$ does not converge absolutely.

(b) $\displaystyle\rho = \lim_{n\to\infty} \left|\frac{(n+1)\cos((n+1)\pi)}{2^{n+1}} \Big/ \frac{n\cos(n\pi)}{2^n}\right| = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1.$ (Note that $\cos(n\pi)$ is just a fancy way of writing $(-1)^n$.) Therefore (ratio test) $\sum_{n=1}^{\infty}((n\cos(n\pi))/2^n)$ converges absolutely.

The Alternating Series Test

We cannot use any of the previously developed tests to show that the alternating harmonic series converges; all of those tests apply only to (ultimately) positive series, so they can test only for absolute convergence. Demonstrating convergence that is not absolute is generally harder to do. We present only one test that can establish such convergence; this test can only be used on a very special kind of series.

THEOREM 14 **The alternating series test**

Suppose that the sequence $\{a_n\}$ is positive, decreasing, and converges to 0, that is, suppose that

(i) $a_n \geq 0$ for $n = 1, 2, 3, \ldots,$

(ii) $a_{n+1} \leq a_n$ for $n = 1, 2, 3, \ldots,$ and

(iii) $\lim_{n\to\infty} a_n = 0$.

Then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

converges.

BEWARE

Read this proof slowly and think about why each statement is true.

PROOF Since the sequence $\{a_n\}$ is decreasing, we have $a_{2n+1} \geq a_{2n+2}$. Therefore $s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n}$ for $n = 1, 2, 3, \ldots$; the even partial sums $\{s_n\}$ form an increasing sequence. Similarly $s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \leq s_{2n-1}$, so the odd partial sums $\{s_{2n-1}\}$ form a decreasing sequence. Since $s_{2n} = s_{2n-1} - a_{2n} \leq s_{2n-1}$, we can say, for any $n \geq 1$, that

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq s_{2n-1} \leq s_{2n-3} \leq \cdots \leq s_5 \leq s_3 \leq s_1.$$

Hence, s_2 is a lower bound for the decreasing sequence $\{s_{2n-1}\}$, and s_1 is an upper bound for the increasing sequence $\{s_{2n}\}$. Both of these sequences therefore converge by the completeness of the real numbers:

$$\lim_{n\to\infty} s_{2n-1} = s_{\text{odd}}, \qquad \lim_{n\to\infty} s_{2n} = s_{\text{even}}.$$

Now $a_{2n} = s_{2n-1} - s_{2n}$, so $0 = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty}(s_{2n-1} - s_{2n}) = s_{\text{odd}} - s_{\text{even}}$. Therefore $s_{\text{odd}} = s_{\text{even}} = s$, say. Every partial sum s_n is either of the form s_{2n-1} or of the form s_{2n}. Thus, $\lim_{n\to\infty} s_n = s$ exists and the series $\sum(-1)^{n-1}a_n$ converges to this sum s.

Remark Note that the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ begins with a positive term, a_1. The conclusion of Theorem 14 also holds for the series $\sum_{n=1}^{\infty}(-1)^n a_n$, which starts with a negative term, $-a_1$. (It is just the negative of the first series.) The theorem also remains valid if the conditions that $\{a_n\}$ is positive and decreasing are replaced by the corresponding *ultimate* versions:

$$\text{(i)} \quad a_n \geq 0 \quad \text{and} \quad \text{(ii)}\ a_{n+1} \leq a_n \quad \text{for } n = N, N+1, N+2, \ldots.$$

Remark The proof of Theorem 14 shows that every even partial sum is less than or equal to s and every odd partial sum is greater than or equal to s. (The reverse is true if $(-1)^n$ is used instead of $(-1)^{n-1}$.) That is, s lies between s_{2n} and either s_{2n-1} or s_{2n+1}. It follows that, for odd or even n,

$$|s - s_n| \leq |s_{n+1} - s_n| = a_{n+1}.$$

This proves the following theorem.

THEOREM 15 **Error estimate for alternating series**

If the sequence $\{a_n\}$ satisfies the conditions of the alternating series test (Theorem 14), so that the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ (or $\sum_{n=1}^{\infty}(-1)^n a_n$) converges to the sum s, then for any $n \geq 1$, the nth partial sum s_n of the series satisfies

$$|s - s_n| \leq |a_{n+1}|.$$

That is, the size of the error involved in using s_n as an approximation to s is less than the size of the first omitted term.

Example 2 How many terms of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{1+2^n}$ are needed to compute the sum of the series with error less than 0.001?

Solution This series satisfies the hypotheses for Theorem 15. If we use the partial sum of the first n terms of the series to approximate the sum of the series, the error will satisfy

$$|\text{error}| \leq |\text{first omitted term}| = \frac{1}{1+2^{n+1}}.$$

This error is less than 0.001 if $1 + 2^{n+1} > 1{,}000$. Since $2^{10} = 1{,}024$, $n + 1 = 10$ will do; we need 9 terms of the series to compute the sum to within 0.001 of its actual value.

When determining the convergence of a given series, it is best to consider first whether the series converges absolutely. If it does not, then there remains the possibility of conditional convergence.

Example 3 Test the following series for absolute and conditional convergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, (b) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln n}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$.

Solution The absolute values of the terms in series (a) and (b) are $1/n$ and $1/(\ln n)$, respectively. Since $1/(\ln n) > 1/n$, and $\sum_{n=1}^{\infty} 1/n$ diverges to infinity, neither series (a) nor (b) converges absolutely. However, both $\{1/n\}$ and $\{1/(\ln n)\}$ are positive, decreasing sequences that converge to 0. Therefore, both (a) and (b) converge by Theorem 14. Each of these series is conditionally convergent.

Series (c) is absolutely convergent because $|(-1)^{n-1}/n^4| = 1/n^4$, and $\sum_{n=1}^{\infty} 1/n^4$ is a convergent p-series ($p = 4 > 1$). We could establish its convergence using Theorem 14, but there is no need to do that since every absolutely convergent series is convergent (Theorem 13).

Example 4 For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\,2^n}$ converge absolutely? converge conditionally? diverge?

Solution For such series whose terms involve functions of a variable x, it is usually wisest to begin testing for absolute convergence with the ratio test. We have

$$\rho = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{(n+1)2^{n+1}} \bigg/ \frac{(x-5)^n}{n\,2^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \left| \frac{x-5}{2} \right| = \left| \frac{x-5}{2} \right|.$$

The series converges absolutely if $|(x-5)/2| < 1$. This inequality is equivalent to $|x-5| < 2$ (the distance from x to 5 is less than 2), that is, $3 < x < 7$. If $x < 3$ or $x > 7$, then $|(x-5)/2| > 1$. The series diverges; its terms do not approach zero.

If $x = 3$, the series is $\sum_{n=1}^{\infty}((-1)^n/n)$, which converges conditionally (it is an alternating harmonic series); if $x = 7$, the series is the harmonic series $\sum_{n=1}^{\infty} 1/n$, which diverges to infinity. Hence, the given series converges absolutely on the open interval $]3, 7[$, converges conditionally at $x = 3$, and diverges everywhere else.

Example 5 For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} (n+1)^2 \left(\frac{x}{x+2}\right)^n$ converge absolutely? converge conditionally? diverge?

Solution Again we begin with the ratio test.

$$\rho = \lim_{n\to\infty} \left| (n+2)^2 \left(\frac{x}{x+2}\right)^{n+1} \bigg/ (n+1)^2 \left(\frac{x}{x+2}\right)^n \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^2 \left|\frac{x}{x+2}\right| = \left|\frac{x}{x+2}\right| = \frac{|x|}{|x+2|}$$

The series converges absolutely if $|x|/|x+2| < 1$. This condition says that the distance from x to 0 is less than the distance from x to -2. Hence $x > -1$. The series diverges if $|x|/|x+2| > 1$, that is, if $x < -1$. If $x = -1$, the series is $\sum_{n=0}^{\infty}(-1)^n(n+1)^2$, which diverges. We conclude that the series converges absolutely for $x > -1$, converges conditionally nowhere, and diverges for $x \le -1$.

When using the alternating series test, it is important to verify (at least mentally) that *all three conditions* (i)–(iii) are satisfied. (As mentioned above, conditions (i) and (ii) need only be satisfied ultimately.)

Example 6 Test the following series for convergence:

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n}$,

(b) $\displaystyle 1-\frac{1}{4}+\frac{1}{3}-\frac{1}{16}+\frac{1}{5}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where

$$a_n=\begin{cases}1/n & \text{if } n \text{ is odd,}\\ 1/n^2 & \text{if } n \text{ is even.}\end{cases}$$

Solution

(a) Here, $a_n = (n+1)/n$ is positive and decreases as n increases. However, $\lim_{n\to\infty} a_n = 1 \neq 0$. The alternating series test does not apply. In fact, the given series diverges because its terms do not approach 0.

(b) This series alternates, a_n is positive, and $\lim_{n\to\infty} a_n = 0$. However, $\{a_n\}$ is not decreasing (even ultimately). Once again, the alternating series test cannot be applied. In fact, since

$$-\frac{1}{4}-\frac{1}{16}-\cdots-\frac{1}{(2n)^2}-\cdots \qquad \text{converges, and}$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}+\cdots \qquad \text{diverges to infinity,}$$

it is readily seen that the given series diverges to infinity. ■

Rearranging the Terms in a Series

The basic difference between absolute and conditional convergence is that when a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, it does so because its terms $\{a_n\}$ decrease in size fast enough that their sum can be finite even if no cancellation occurs due to terms of opposite sign. If cancellation is required to make the series converge (because the terms decrease slowly), then the series can only converge conditionally.

Consider the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots.$$

This series converges, but only conditionally. If we take the subseries containing only the positive terms, we get the series

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots,$$

which diverges to infinity. Similarly, the subseries of negative terms

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots$$

diverges to negative infinity.

If a series converges absolutely, the subseries consisting of positive terms and the subseries consisting of negative terms must each converge to a finite sum. If a series converges conditionally, the positive and negative subseries will both diverge, to ∞ and $-\infty$, respectively.

Using these facts we can answer a question raised at the beginning of Section 9.2. If we rearrange the terms of a convergent series so that they are added in a different order, must the rearranged series converge or not, and if it does will it converge to the same sum? The answer depends on whether the original series was absolutely convergent or merely conditionally convergent.

THEOREM 16 **Convergence of rearrangements of a series**

(a) If the terms of an absolutely convergent series are rearranged so that addition occurs in a different order, the rearranged series still converges to the same sum as the original series.

(b) If a series is conditionally convergent, and L is any real number, then the terms of the series can be rearranged so as to make the series converge (conditionally) to the sum L. It can also be rearranged so as to diverge to ∞ or to $-\infty$, or just to diverge.

Part (b) shows that conditional convergence is a rather suspect kind of convergence, being dependent on the order in which the terms are added. We will not present a formal proof of the theorem but will give an example suggesting what is involved. (See also Exercise 30 below.)

Example 7 In Section 9.5 we will show that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

converges (conditionally) to the sum $\ln 2$. Describe how to rearrange its terms so that it converges to 8 instead.

Solution Start adding terms of the positive subseries

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots,$$

and keep going until the partial sum exceeds 8. (It will, eventually, because the positive subseries diverges to infinity.) Then add the first term $-1/2$ of the negative subseries

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots.$$

This will reduce the partial sum below 8 again. Now resume adding terms of the positive subseries until the partial sum climbs above 8 once more. Then add the second term of the negative subseries and the partial sum will drop below 8.

Keep repeating this procedure, alternately adding terms of the positive subseries to force the sum above 8 and then terms of the negative subseries to force it below 8. Since both subseries have infinitely many terms and diverge to ∞ and $-\infty$, respectively, eventually every term of the original series will be included, and the partial sums of the new series will oscillate back and forth around 8, converging to that number. Of course, any number other than 8 could also be used in place of 8. ■

Exercises 9.4

Determine whether the series in Exercises 1–12 converge absolutely, converge conditionally, or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{(n+1)\ln(n+1)}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2^n}$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n^2-1)}{n^2+1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi^n}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{-n}{n^2+1}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{20n^2 - n - 1}{n^3 + n^2 + 33}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-100)^n}$

12. $\displaystyle\sum_{n=10}^{\infty} \frac{\sin(n+1/2)\pi}{\ln\ln n}$

For the series in Exercises 13–16, find the smallest integer n that ensures that the partial sum s_n approximates the sum s of the series with error less than 0.001 in absolute value.

13. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

14. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

15. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2^n}$

16. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!}$

Determine the values of x for which the series in Exercises 17–24 converge absolutely, converge conditionally, or diverge.

17. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 2^{2n}}$

19. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2n+3}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{2^n \ln n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/3}4^n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^n$

∗ **25.** Does the alternating series test apply directly to the series $\sum_{n=1}^{\infty}(1/n)\sin(n\pi/2)$? Determine whether the series converges.

∗ **26.** Show that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $a_n = 10/n^2$ for even n and $a_n = -1/10n^3$ for odd n.

∗ **27.** Which of the following statements are TRUE and which are FALSE? Justify your assertion of truth, or give a counterexample to show falsehood.

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty}(-1)^n a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(c) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely.

∗ **28.** (a) Use a Riemann sum argument to show that

$$\ln n! \geq \int_1^n \ln t\,dt = n\ln n - n + 1.$$

(b) For what values of x does the series $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$ converge absolutely? converge conditionally? diverge? (*Hint:* First use the ratio test. To test the cases where $\rho = 1$, you may find the inequality in part (a) useful.)

∗ **29.** For what values of x does the series $\sum_{n=1}^{\infty} \frac{(2n)!x^n}{2^{2n}(n!)^2}$ converge absolutely? converge conditionally? diverge? *Hint:* see Exercise 42 of Section 9.3.

∗ **30.** Devise procedures for rearranging the terms of the alternating harmonic series so that the rearranged series (a) diverges to ∞, (b) converges to -2.

9.5 Power Series

This section is concerned with a special kind of infinite series called a *power series*, which may be thought of as a polynomial of infinite degree.

DEFINITION 7

Power series

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

is called a **power series in powers of** $x-c$ or a **power series about the point** $x = c$. The constants $a_0, a_1, a_2, \ldots$ are called the **coefficients** of the power series.

Since the terms of a power series are functions of a variable x, the series may or may not converge for each value of x. For those values of x for which the series does converge, the sum defines a function of x. For example, if $-1 < x < 1$, then

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

The geometric series on the left side is a power series *representation* of the function $1/(1-x)$ in powers of x (or about the point $x = 0$). Note that the representation is valid only in the open interval $]-1, 1[$ even though $1/(1-x)$ is defined for all real x except $x = 1$. For $x = -1$ and for $|x| > 1$ the series does not converge, so it cannot represent $1/(1-x)$ at these points.

The point c is the **centre of convergence** of the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. The series certainly converges (to a_0) at $x = c$. (All the terms except possibly the first are 0.) Theorem 17 below shows that if the series converges anywhere else, then it converges on an interval (possibly infinite) centred at $x = c$, and it converges absolutely everywhere on that interval except possibly at one or both of the endpoints if the interval is finite. The geometric series

$$1 + x + x^2 + x^3 + \cdots$$

is an example of this behaviour. It has centre of convergence $c = 0$, and converges only on the interval $]-1, 1[$, centred at 0. The convergence is absolute at every point of the interval. Another example is the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}(x-5)^n = \frac{x-5}{2} + \frac{(x-5)^2}{2 \times 2^2} + \frac{(x-5)^3}{3 \times 2^3} + \cdots,$$

which we discussed in Example 4 of Section 9.4. We showed that this series converges on the interval $[3, 7[$, an interval with centre $x = 5$, and that the convergence is absolute on the open interval $]3, 7[$ but is only conditional at the endpoint $x = 3$.

THEOREM 17

For any power series $\sum_{n=0}^{\infty} a_n\,(x-c)^n$ one of the following alternatives must hold:

(i) the series may converge only at $x = c$,

(ii) the series may converge at every real number x, or

(iii) there may exist a positive real number R such that the series converges at every x satisfying $|x - c| < R$ and diverges at every x satisfying $|x - c| > R$. In this case the series may or may not converge at either of the two *endpoints* $x = c - R$ and $x = c + R$.

In each of these cases the convergence is absolute except possibly at the endpoints $x = c - R$ and $x = c + R$ in case (iii).

PROOF We observed above that every power series converges at its centre of convergence; only the first term can be nonzero so the convergence is absolute. To prove the rest of this theorem, it suffices to show that if the series converges at any number $x_0 \neq c$, then it converges absolutely at every number x closer to c than x_0 is, that is, at every x satisfying $|x - c| < |x_0 - c|$. This means that convergence at any $x_0 \neq c$ implies absolute convergence on $]c - x_0, c + x_0[$, so the set of points x where the series converges must be an interval centred at c.

Suppose, therefore, that $\sum_{n=0}^{\infty} a_n(x_0 - c)^n$ converges. Then $\lim a_n(x_0 - c)^n = 0$, so $|a_n(x_0 - c)^n| \leq K$ for all n, where K is some constant (Theorem 1 of Section 9.1). If $r = |x - c|/|x_0 - c| < 1$, then

$$\sum_{n=0}^{\infty} |a_n(x - c)^n| = \sum_{n=0}^{\infty} |a_n(x_0 - c)^n| \left|\frac{x - c}{x_0 - c}\right|^n \leq K \sum_{n=0}^{\infty} r^n = \frac{K}{1 - r} < \infty.$$

Thus $\sum_{n=0}^{\infty} a_n(x - c)^n$ converges absolutely.

By Theorem 17, the set of values x for which the power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ converges is an interval centred at $x = c$. We call this interval the **interval of convergence** of the power series. It must be of one of the following forms:

(i) the isolated point $x = c$ (a degenerate closed interval $[c, c]$),

(ii) the entire line $]-\infty, \infty[$,

(iii) a finite interval centred at c:
$[c - R, c + R]$, or $[c - R, c + R[$, or $]c - R, c + R]$, or $]c - R, c + R[$.

The number R in (iii) is called the **radius of convergence** of the power series. In case (i) we say the radius of convergence is $R = 0$; in case (ii) it is $R = \infty$.

The radius of convergence, R, can often be found by using the ratio test on the power series: if

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}(x - c)^{n+1}}{a_n(x - c)^n}\right| = \left(\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\right) |x - c|$$

exists, then the series $\sum_{n=0}^{\infty} a_n(x - c)^n$ converges absolutely where $\rho < 1$, that is, where

$$|x - c| < R = 1 \Big/ \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

The series diverges if $|x - c| > R$.

Radius of convergence

Suppose that $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞. Then the power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ has radius of convergence $R = 1/L$. (If $L = 0$, then $R = \infty$; if $L = \infty$, then $R = 0$.)

Example 1 Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}.$$

Solution The series can be rewritten

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1} \left(x+\frac{5}{2}\right)^n.$$

The centre of convergence is $x = -5/2$. The radius of convergence, R, is given by

$$\frac{1}{R} = L = \lim \left| \frac{\left(\frac{2}{3}\right)^{n+1} \frac{1}{(n+1)^2+1}}{\left(\frac{2}{3}\right)^n \frac{1}{n^2+1}} \right| = \lim \frac{2}{3} \frac{n^2+1}{(n+1)^2+1} = \frac{2}{3}.$$

Thus $R = 3/2$. The series converges absolutely on $]-5/2-3/2, -5/2+3/2[=]-4,-1[$, and it diverges on $]-\infty,-4[$ and on $]-1,\infty[$. At $x=-1$ the series is $\sum_{n=0}^{\infty} 1/(n^2+1)$; at $x=-4$ it is $\sum_{n=0}^{\infty}(-1)^n/(n^2+1)$. Both series converge (absolutely). The interval of convergence of the given power series is therefore $[-4,-1]$. ■

Example 2 Determine the radii of convergence of the series

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and (b) $\displaystyle\sum_{n=0}^{\infty} n!x^n$.

Solution

(a) $L = \left| \lim \frac{1}{(n+1)!} \Big/ \frac{1}{n!} \right| = \lim \frac{n!}{(n+1)!} = \lim \frac{1}{n+1} = 0$. Thus $R = \infty$.
This series converges (absolutely) for all x. The sum is e^x, as will be shown in Example 1 in the next section.

(b) $L = \left| \lim \frac{(n+1)!}{n!} \right| = \lim(n+1) = \infty$. Thus $R = 0$
This series converges only at its centre of convergence, $x = 0$. ■

Algebraic Operations on Power Series

To simplify the following discussion, we will consider only power series with $x = 0$ as centre of convergence, that is, series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Any properties we demonstrate for such series extend automatically to power series of the form $\sum_{n=0}^{\infty} a_n (y-c)^n$ via the change of variable $x = y - c$.

First we observe that series having the same centre of convergence can be added or subtracted on whatever interval is common to their intervals of convergence. The following theorem is a simple consequence of Theorem 7 of Section 9.2 and does not require a proof.

THEOREM 18 Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series with radii of convergence R_a and R_b, respectively, and let c be a constant. Then

(i) $\sum_{n=0}^{\infty}(ca_n)\, x^n$ has radius of convergence R_a, and

$$\sum_{n=0}^{\infty}(ca_n)\, x^n = c \sum_{n=0}^{\infty} a_n\, x^n$$

wherever the series on the right converges.

(ii) $\sum_{n=0}^{\infty}(a_n + b_n)\, x^n$ has radius of convergence R at least as large as the smaller of R_a and R_b ($R \geq \min\{R_a, R_b\}$), and

$$\sum_{n=0}^{\infty}(a_n + b_n)\, x^n = \sum_{n=0}^{\infty} a_n\, x^n + \sum_{n=0}^{\infty} b_n\, x^n$$

wherever both series on the right converge.

The situation regarding multiplication and division of power series is more complicated. We will mention only the results and will not attempt any proofs of our assertions. A textbook in mathematical analysis will provide more details.

Long multiplication of the form

$$\begin{aligned}(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots)\\ = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots\end{aligned}$$

leads us to conjecture the formula

$$\left(\sum_{n=0}^{\infty} a_nx^n\right)\left(\sum_{n=0}^{\infty} b_nx^n\right) = \sum_{n=0}^{\infty} c_nx^n,$$

where

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0 = \sum_{j=0}^{n} a_jb_{n-j}.$$

The series $\sum_{n=0}^{\infty} c_nx^n$ is called the **Cauchy product** of the series $\sum_{n=0}^{\infty} a_nx^n$ and $\sum_{n=0}^{\infty} b_nx^n$. Like the sum, the Cauchy product also has radius of convergence at least equal to the lesser of those of the factor series.

Example 3 Since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

holds for $-1 < x < 1$, we can determine a power series representation for $1/(1-x)^2$

by taking the Cauchy product of this series with itself. Since $a_n = b_n = 1$ for $n = 0, 1, 2, \ldots$, we have

$$c_n = \sum_{j=0}^{n} 1 = n + 1$$

and

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n,$$

which must also hold for $-1 < x < 1$. The same series can be obtained by direct long multiplication of the series:

$$\begin{array}{rccccccccc}
 & 1 & + & x & + & x^2 & + & x^3 & + & \cdots \\
\times & 1 & + & x & + & x^2 & + & x^3 & + & \cdots \\
\hline
 & 1 & + & x & + & x^2 & + & x^3 & + & \cdots \\
 & & & x & + & x^2 & + & x^3 & + & \cdots \\
 & & & & & x^2 & + & x^3 & + & \cdots \\
 & & & & & & & x^3 & + & \cdots \\
 & & & & & & & & & \cdots \\
\hline
 & 1 & + & 2x & + & 3x^2 & + & 4x^3 & + & \cdots
\end{array}$$

■

Long division can also be performed on power series, but there is no simple rule for determining the coefficients of the quotient series. The radius of convergence of the quotient series is not less than the least of the three numbers R_1, R_2, and R_3, where R_1 and R_2 are the radii of convergence of the divisor and dividend series and R_3 is the distance from the centre of convergence to the nearest complex number where the divisor series has sum equal to 0. To illustrate this point, observe that 1 and $1 - x$ are both power series with infinite radii of convergence:

$$\begin{aligned} 1 &= 1 + 0x + 0x^2 + 0x^3 + \cdots && \text{for all } x, \\ 1 - x &= 1 - \ x + 0x^2 + 0x^3 + \cdots && \text{for all } x. \end{aligned}$$

Their quotient, $1/(1-x)$, however, only has radius of convergence 1, the distance from the centre of convergence $x = 0$ to the point $x = 1$ where the denominator vanishes:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{for} \quad |x| < 1.$$

Differentiation and Integration of Power Series

If a power series has a positive radius of convergence, it can be differentiated or integrated term by term. The resulting series will converge to the appropriate derivative or integral of the sum of the original series everywhere except possibly at the endpoints of the interval of convergence of the original series. This very important fact ensures that, for purposes of calculation, power series behave just like polynomials, the easiest functions to differentiate and integrate. We formalize the differentiation and integration properties of power series in the following theorem.

THEOREM 19 **Term by term differentiation and integration of power series**

If the series $\sum_{n=0}^{\infty} a_n x^n$ converges to the sum $f(x)$ on an interval $]-R, R[$, where $R > 0$, that is,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \qquad (-R < x < R),$$

then f is differentiable on $]-R, R[$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots, \qquad (-R < x < R).$$

Also, f is integrable over any closed subinterval of $]-R, R[$, and if $|x| < R$, then

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots.$$

PROOF Let x satisfy $-R < x < R$ and choose $H > 0$ such that $|x| + H < R$. By Theorem 17 we then have[1]

$$\sum_{n=1}^{\infty} |a_n|(|x| + H)^n = K < \infty.$$

The Binomial Theorem (see Section 9.9) shows that if $n \geq 1$, then

$$(x+h)^n = x^n + n x^{n-1} h + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k.$$

Therefore, if $|h| \leq H$ we have

$$\begin{aligned} |(x+h)^n - x^n - n x^{n-1} h| &= \left| \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k \right| \\ &\leq \sum_{k=2}^{n} \binom{n}{k} |x|^{n-k} \frac{|h|^k}{H^k} H^k \\ &\leq \frac{|h|^2}{H^2} \sum_{k=0}^{n} \binom{n}{k} |x|^{n-k} H^k \\ &= \frac{|h|^2}{H^2} (|x| + H)^n. \end{aligned}$$

Also

$$|n x^{n-1}| = \frac{n |x|^{n-1} H}{H} \leq \frac{1}{H} (|x| + H)^n.$$

Thus

$$\sum_{n=1}^{\infty} |n a_n x^{n-1}| \leq \frac{1}{H} \sum_{n=1}^{\infty} |a_n|(|x| + H)^n = \frac{K}{H} < \infty,$$

[1] This proof is due to R. Výborný, *American Mathematical Monthly*, April 1987.

so the series $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges (absolutely) to $g(x)$, say. Now

$$\begin{aligned}\left|\frac{f(x+h)-f(x)}{h}-g(x)\right| &= \left|\sum_{n=1}^{\infty}\frac{a_n(x+h)^n - a_n x^n - na_n x^{n-1}h}{h}\right| \\ &\le \frac{1}{|h|}\sum_{n=1}^{\infty}|a_n||(x+h)^n - x^n - nx^{n-1}h| \\ &\le \frac{|h|}{H^2}\sum_{n=1}^{\infty}|a_n|\big(|x|+H\big)^n \le \frac{K|h|}{H^2}.\end{aligned}$$

Letting h approach zero, we obtain $|f'(x) - g(x)| \le 0$, so $f'(x) = g(x)$, as required.

Now observe that since $|a_n/(n+1)| \le |a_n|$, the series

$$h(x) = \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$$

converges (absolutely) at least on the interval $]-R, R[$. Using the differentiation result proved above, we obtain

$$h'(x) = \sum_{n=0}^{\infty} a_n x^n = f(x).$$

Since $h(0) = 0$, we have

$$\int_0^x f(t)\,dt = \int_0^x h'(t)\,dt = h(t)\Big|_0^x = h(x),$$

as required.

Together, these results imply that the termwise differentiated or integrated series have the same radius of convergence as the given series. In fact, as the following examples illustrate, the interval of convergence of the differentiated series is the same as that of the original series except for the *possible* loss of one or both endpoints if the original series converges at endpoints of its interval of convergence. Similarly, the integrated series will converge everywhere on the interval of convergence of the original series and possibly at one or both endpoints of that interval, even if the original series does not converge at the endpoints.

Being differentiable on $]-R, R[$, where R is the radius of convergence, the sum $f(x)$ of a power series is necessarily continuous on that open interval. If the series happens to converge at either or both of the endpoints $-R$ and R, then f is also continuous (on one side) up to these endpoints. This result is stated formally in the following theorem. We will not prove it here; the interested reader is referred to textbooks on mathematical analysis for a proof.

THEOREM 20 **Abel's Theorem**

The sum of a power series is a continuous function everywhere on the interval of convergence of the series. In particular, if $\sum_{n=0}^{\infty} a_n R^n$ converges for some $R > 0$, then

$$\lim_{x\to R-}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n,$$

and if $\sum_{n=0}^{\infty} a_n(-R)^n$ converges, then

$$\lim_{x \to -R+} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n(-R)^n.$$

The following examples show how the above theorems are applied to obtain power series representations for functions.

Example 4 Find power series representations for the functions

(a) $\dfrac{1}{(1-x)^2}$, (b) $\dfrac{1}{(1-x)^3}$, and (c) $\ln(1+x)$

by starting with the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$$

and using differentiation, integration, and substitution. Where is each series valid?

Solution

(a) Differentiate the geometric series term by term to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad (-1 < x < 1).$$

This is the same result obtained by multiplication of series in Example 3 above.

(b) Differentiate again to get, for $-1 < x < 1$,

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)\, x^{n-2} = (1 \times 2) + (2 \times 3)x + (3 \times 4)x^2 + \cdots.$$

Now divide by 2:

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = 1 + 3x + 6x^2 + 10x^3 + \cdots \qquad (-1 < x < 1).$$

(c) Substitute $-t$ in place of x in the original geometric series:

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n = 1 - t + t^2 - t^3 + t^4 - \cdots \qquad (-1 < t < 1).$$

Integrate from 0 to x, where $|x| < 1$, to get

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\, dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (-1 < x \le 1). \end{aligned}$$

Note that the latter series converges (conditionally) at the endpoint $x = 1$ as well as on the interval $-1 < x < 1$. Since $\ln(1 + x)$ is continuous at $x = 1$, Theorem 20 assures us that the series must converge to that function at $x = 1$ also. In particular, therefore, the alternating harmonic series converges to $\ln 2$:

Example 5 Use the geometric series of the previous example to find a power series representation for $\tan^{-1} x$.

Solution Substitute $-t^2$ for x in the geometric series. Since $0 \le t^2 < 1$ whenever $-1 < t < 1$, we obtain

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots \qquad (-1 < t < 1).$$

Now integrate from 0 to x, where $|x| < 1$:

$$\begin{aligned}\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} &= \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \cdots)\, dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (-1 < x < 1).\end{aligned}$$

However, note that the series also converges (conditionally) at $x = -1$ and 1. Since $\tan^{-1}$ is continuous at ± 1, the above series representation for $\tan^{-1} x$ also holds for these values, by Theorem 20. Letting $x = 1$ we get another interesting result:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

Again, however, this would not be a good formula with which to calculate a numerical value of π. (Why not?)

Example 6 Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots.$$

Solution Observe (in Example 4(a)) how the process of differentiating the geometric series produces a series with coefficients 1, 2, 3, ... Start with the series obtained for $1/(1-x)^2$ and multiply it by x to obtain

$$\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2}.$$

Now differentiate again to get a series with coefficients 1^2, 2^2, 3^2, ...:

$$\sum_{n=1}^{\infty} n^2x^{n-1} = 1 + 4x + 9x^2 + 16x^3 + \cdots = \frac{d}{dx}\frac{x}{(x-1)^2} = \frac{1+x}{(1-x)^3}.$$

Multiplication by x again gives the desired power series:

$$\sum_{n=1}^{\infty} n^2x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots = \frac{x(1+x)}{(1-x)^3}.$$

Differentiation and multiplication by x do not change the radius of convergence, so this series converges to the indicated function for $-1 < x < 1$. Putting $x = 1/2$, we get

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2} \times \frac{3}{2}}{\frac{1}{8}} = 6.$$

■

The following example illustrates how substitution can be used to obtain power series representations of functions with centres of convergence different from 0.

Example 7 Find a series representation of $f(x) = 1/(2+x)$ in powers of $x - 1$. What is the interval of convergence of this series?

Solution Let $t = x - 1$ so that $x = t + 1$. We have

$$\begin{aligned}
\frac{1}{2+x} &= \frac{1}{3+t} = \frac{1}{3}\frac{1}{1+\frac{t}{3}} \\
&= \frac{1}{3}\left(1 - \frac{t}{3} + \frac{t^2}{3^2} - \frac{t^3}{3^3} + \cdots\right) && (-1 < t/3 < 1) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{3^{n+1}} && (-3 < t < 3) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{3^{n+1}} && (-2 < x < 4).
\end{aligned}$$

Note that the radius of convergence of this series is 3, the distance from the centre of convergence, 1, to the point -2 where the denominator is 0. We could have predicted this in advance.

■

Maple Calculations

Maple can find the sums of many kinds of series, including absolutely and conditionally convergent numerical series and many power series. Even when Maple can't find the formal sum of a (convergent) series, it can provide a decimal approximation to the precision indicated by the current value of its variable `Digits`, which defaults to 10. Here are some examples.

```
> sum(n^4/2^n, n=1..infinity);
```

$$150$$

```
> sum(1/n^2, n=1..infinity);
```

$$\frac{1}{6}\pi^2$$

```
> sum(exp(-n^2), n=0..infinity);
```

$$\sum_{n=0}^{\infty} e^{(-n^2)}$$

```
> evalf(%);
```

$$1.386318602$$

```
> f := x -> sum(x^(n-1)/n, n=1..infinity);
```

$$f := x \to \sum_{n=1}^{\infty} \frac{x^{(n-1)}}{n}$$

```
> f(1); f(-1); f(1/2);
```

$$\infty$$
$$\ln(2)$$
$$2\ln(2)$$

Exercises 9.5

Determine the centre, radius, and interval of convergence of each of the power series in Exercises 1–8.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

2. $\displaystyle\sum_{n=0}^{\infty} 3n\,(x+1)^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x+2}{2}\right)^n$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}\,x^n$

5. $\displaystyle\sum_{n=0}^{\infty} n^3(2x-3)^n$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}\,(4-x)^n$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(1+5^n)}{n!}\,x^n$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

9. Use multiplication of series to find a power series representation of $1/(1-x)^3$ valid in the interval $]-1, 1[$.

10. Determine the Cauchy product of the series $1 + x + x^2 + x^3 + \cdots$ and $1 - x + x^2 - x^3 + \cdots$. On what interval and to what function does the product series converge?

11. Determine the power series expansion of $1/(1-x)^2$ by formally dividing $1 - 2x + x^2$ into 1.

Starting with the power series representation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad (-1 < x < 1),$$

determine power series representations for the functions indicated in Exercises 12–20. On what interval is each representation valid?

12. $\dfrac{1}{2-x}$ in powers of x

13. $\dfrac{1}{(2-x)^2}$ in powers of x

14. $\dfrac{1}{1+2x}$ in powers of x

15. $\ln(2-x)$ in powers of x

16. $\dfrac{1}{x}$ in powers of $x-1$

17. $\dfrac{1}{x^2}$ in powers of $x+2$

18. $\dfrac{1-x}{1+x}$ in powers of x

19. $\dfrac{x^3}{1-2x^2}$ in powers of x

20. $\ln x$ in powers of $x-4$

Determine the interval of convergence and the sum of each of the series in Exercises 21–26.

21. $1 - 4x + 16x^2 - 64x^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n (4x)^n$

*22. $3 + 4x + 5x^2 + 6x^3 + \cdots = \sum_{n=0}^{\infty}(n+3)x^n$

*23. $\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n+3}$

*24. $1 \times 3 - 2 \times 4x + 3 \times 5x^2 - 4 \times 6x^3 + \cdots$
$= \sum_{n=0}^{\infty}(-1)^n(n+1)(n+3)\,x^n$

*25. $2 + 4x^2 + 6x^4 + 8x^6 + 10x^8 + \cdots = \sum_{n=0}^{\infty}2(n+1)\,x^{2n}$

*26. $1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n+1}$

Use the technique (or the result) of Example 6 to find the sums of the numerical series in Exercises 27–32.

27. $\sum_{n=1}^{\infty}\frac{n}{3^n}$

28. $\sum_{n=0}^{\infty}\frac{n+1}{2^n}$

*29. $\sum_{n=0}^{\infty}\frac{(n+1)^2}{\pi^n}$

*30. $\sum_{n=1}^{\infty}\frac{(-1)^n n(n+1)}{2^n}$

31. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n}$

32. $\sum_{n=3}^{\infty}\frac{1}{n2^n}$

9.6 Taylor and Maclaurin Series

If a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has a positive radius of convergence R, then the sum of the series defines a function $f(x)$ on the interval $]c-R, c+R[$. We say that the power series is a **representation** of $f(x)$ on that interval. What relationship exists between the function $f(x)$ and the coefficients $a_0, a_1, a_2, \ldots$ of the power series? The following theorem answers this question.

THEOREM 21 Suppose the series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

converges to $f(x)$ for $c - R < x < c + R$, where $R > 0$. Then

$$a_k = \frac{f^{(k)}(c)}{k!} \qquad \text{for } k = 0, 1, 2, 3, \ldots.$$

PROOF This proof requires that we differentiate the series for $f(x)$ term by term several times, a process justified by Theorem 19 (suitably reformulated for powers of $x - c$):

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2} = 2a_2 + 6a_3(x-c) + 12a_4(x-c)^2 + \cdots$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n(x-c)^{n-k}$$
$$= k!a_k + \frac{(k+1)!}{1!}a_{k+1}(x-c) + \frac{(k+2)!}{2!}a_{k+2}(x-c)^2 + \cdots.$$

Each series converges for $c - R < x < c + R$. Setting $x = c$, we obtain $f^{(k)}(c) = k!a_k$, which proves the theorem.

Theorem 21 shows that a function $f(x)$ that has a power series representation with centre at c and positive radius of convergence must have derivatives of all orders in an interval around $x = c$, and it can have only one representation as a power series in powers of $x - c$, namely

$$f(x) = \sum^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

DEFINITION 8

Taylor and Maclaurin series

If $f(x)$ has derivatives of all orders at $x = c$ (i.e., if $f^{(k)}(c)$ exists for $k = 0, 1, 2, 3, \ldots$), then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$
$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \cdots$$

is called the **Taylor series of f about $x = c$** (or the **Taylor series of f in powers of $x - c$**). If $c = 0$, the term **Maclaurin series** is usually used in place of Taylor series.

Note that the partial sums of such Taylor (or Maclaurin) series are just the Taylor (or Maclaurin) polynomials studied in Section 4.8.

The Taylor series is a power series as defined in the previous section. Theorem 17 implies that c must be the centre of any interval on which such a series converges, but the definition of Taylor series makes no requirement that the series should converge anywhere except at the point $x = c$ where the series is just $f(0) + 0 + 0 + \cdots$. The series exists provided all the derivatives of f exist at $x = c$; in practice this means that each derivative must exist in an open interval containing $x = c$. (Why?) However, the series may converge nowhere except at $x = c$, and if it does converge elsewhere, it may converge to something other than $f(x)$. (See Exercise 40 at the end of this section for an example where this happens.) If the Taylor series does converge to $f(x)$ in an open interval containing c, then we will say that f is analytic at $x = c$.

DEFINITION 9

Analytic functions

A function $f(x)$ is **analytic** at $x = c$ if $f(x)$ is the sum of a power series in powers of $x - c$ having positive radius of convergence. (The series is its Taylor series.) If f is analytic at each point of an open interval I, then we say it is analytic on the interval I.

Most, but not all, of the elementary functions encountered in calculus are analytic wherever they have derivatives of all orders. On the other hand, whenever a power series converges on an open interval containing c, then its sum $f(x)$ is analytic at c, and the given series is the Taylor series of $f(x)$ about $x = c$.

Maclaurin Series for Some Elementary Functions

Calculating Taylor and Maclaurin series for a function f directly from Definition 8 is practical only when we can find a formula for the nth derivative of f. Examples of such functions include $(ax+b)^r$, e^{ax+b}, $\ln(ax+b)$, $\sin(ax+b)$, $\cos(ax+b)$, and sums of such functions.

Example 1 Find the Taylor series for e^x about $x = c$. Where does the series converge to e^x? Where is e^x analytic? What is the Maclaurin series for e^x?

Solution Since all the derivatives of $f(x) = e^x$ are e^x, we have $f^{(n)}(c) = e^c$ for every integer $n \geq 0$. Thus the Taylor series for e^x about $x = c$ is

$$\sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \cdots.$$

The radius of convergence R of this series is given by

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{e^c/(n+1)!}{e^c/n!} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Thus the radius of convergence is $R = \infty$ and the series converges for all x. Suppose the sum is $g(x)$:

$$g(x) = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \cdots.$$

By Theorem 19, we have

$$\begin{aligned} g'(x) &= 0 + e^c + \frac{e^c}{2!}2(x-c) + \frac{e^c}{3!}3(x-c)^2 + \cdots \\ &= e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \cdots = g(x). \end{aligned}$$

Also, $g(c) = e^c + 0 + 0 + \cdots = e^c$. Since $g(x)$ satisfies the differential equation $g'(x) = g(x)$ of exponential growth, we have $g(x) = Ce^x$. Substituting $x = c$ gives $e^c = g(c) = Ce^c$, so $C = 1$. Thus the Taylor series for e^x in powers of $x - c$ converges to e^x for every real number x:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n \\ &= e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \cdots \qquad \text{(for all } x\text{)}. \end{aligned}$$

In particular, setting $c = 0$ we obtain the Maclaurin series for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{(for all } x\text{)}.$$

■

Example 2 Find the Maclaurin series for (a) $\sin x$ and (b) $\cos x$. Where does each series converge?

Solution Let $f(x) = \sin x$. Then we have $f(0) = 0$ and

$$\begin{aligned} f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \\ &\vdots & &\vdots \end{aligned}$$

Thus, the Maclaurin series for $\sin x$ is

$$\begin{aligned} g(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \end{aligned}$$

We have denoted the sum by $g(x)$ since we don't yet know whether the series converges to $\sin x$. The series does converge for all x by the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}}{(2(n+1)+1)!} x^{2(n+1)+1}}{\dfrac{(-1)^n}{(2n+1)!} x^{2n+1}} \right| &= \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!} |x|^2 \\ &= \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0. \end{aligned}$$

Now we can differentiate the function $g(x)$ twice to get

$$\begin{aligned} g'(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ g''(x) &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots = -g(x). \end{aligned}$$

Thus, $g(x)$ satisfies the differential equation $g''(x) + g(x) = 0$ of simple harmonic motion. The general solution of this equation, as observed in Section 3.7, is

$$g(x) = A \cos x + B \sin x.$$

Observe, from the series, that $g(0) = 0$ and $g'(0) = 1$. These values determine that $A = 0$ and $B = 1$. Thus, $g(x) = \sin x$ and $g'(x) = \cos x$ for all x.

We have therefore demonstrated that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{(for all } x\text{)},$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{(for all } x\text{)}.$$

Theorem 21 shows that we can use any available means to find a power series converging to a given function on an interval, and the series obtained will turn out to be the Taylor series. In Section 9.5 several series were constructed by manipulating a geometric series. These include:

Some Maclaurin series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad (-1 < x < 1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (-1 < x \le 1)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1 \le x \le 1)$$

These series, together with the intervals on which they converge, are frequently used hereafter and should be memorized.

Other Maclaurin and Taylor Series

Series can be combined in various ways to generate new series. For example, we can find the Maclaurin series for e^{-x} by replacing x with $-x$ in the Maclaurin series for e^x:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \qquad \text{(for all } x\text{)}.$$

The series for e^x and e^{-x} can then be subtracted or added and the results divided by 2 to obtain Maclaurin series for the hyperbolic functions $\sinh x$ and $\cosh x$:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{(for all } x\text{)}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{(for all } x\text{)}.$$

Remark Observe the similarity between the series for $\sin x$ and $\sinh x$ and between those for $\cos x$ and $\cosh x$. If we were to allow complex numbers (numbers of the form $z = x + iy$, where $i^2 = -1$ and x and y are real; see Appendix I) as arguments for our functions, and if we were to demonstrate that our operations on series could be extended to series of complex numbers, we would see that $\cos x = \cosh(ix)$ and $\sin x = -i \sinh(ix)$. In fact,

$$e^{ix} = \cos x + i \sin x \qquad \text{and} \qquad e^{-ix} = \cos x - i \sin x,$$

so

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Such formulas are encountered in the study of functions of a complex variable; from the complex point of view the trigonometric and exponential functions are just different manifestations of the same basic function, a complex exponential $e^z = e^{x+iy}$. We content ourselves here with having mentioned the interesting relationships above and invite the reader to verify them formally by calculating with series. (Such formal calculations do not, of course, constitute a proof, since we have not established the various rules covering series of complex numbers.)

(a) We substitute $-x^2/3$ for x in the Maclaurin series for e^x:

$$e^{-x^2/3} = 1 - \frac{x^2}{3} + \frac{1}{2!}\left(\frac{x^2}{3}\right)^2 - \frac{1}{3!}\left(\frac{x^2}{3}\right)^3 + \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{3^n n!} x^{2n} \qquad \text{(for all real } x\text{)}.$$

(b) For all $x \neq 0$ we have

$$\frac{\sin x^2}{x} = \frac{1}{x}\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right)$$
$$= x - \frac{x^5}{3!} + \frac{x^9}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(2n+1)!}.$$

Note that $f(x) = (\sin(x^2))/x$ is not defined at $x = 0$ but does have a limit (namely 0) as x approaches 0. If we define $f(0) = 0$ (the continuous extension of $f(x)$ to $x = 0$), then the series converges to $f(x)$ for all x.

(c) We use a trigonometric identity to express $\sin^2 x$ in terms of $\cos 2x$ and then use the Maclaurin series for $\cos x$ with x replaced by $2x$.

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)$$
$$= \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \cdots\right)$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+2)!} x^{2n+2} \qquad \text{(for all real } x\text{)}.$$

■

Taylor series about points other than 0 can often be obtained from known Maclaurin series by a change of variable.

Example 4 Find the Taylor series for $\ln x$ in powers of $x - 2$. Where does the series converge to $\ln x$?

Solution Note that if $t = (x-2)/2$, then

$$\ln x = \ln(2 + (x-2)) = \ln\left[2\left(1 + \frac{x-2}{2}\right)\right] = \ln 2 + \ln(1+t).$$

We use the known Maclaurin series for $\ln(1+t)$:

$$\begin{aligned}\ln x &= \ln 2 + \ln(1+t)\\ &= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} - \cdots\\ &= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2\times 2^2} + \frac{(x-2)^3}{3\times 2^3} - \frac{(x-2)^4}{4\times 2^4} + \cdots\\ &= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}(x-2)^n.\end{aligned}$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < (x-2)/2 \le 1$, that is, for $0 < x \le 4$. ■

Example 5 Find the Taylor series for $\cos x$ about the point $x = \pi/3$. Where is the series valid?

Solution We use the addition formula for cosine:

$$\begin{aligned}\cos x &= \cos\left(x - \frac{\pi}{3} + \frac{\pi}{3}\right) = \cos\left(x - \frac{\pi}{3}\right)\cos\frac{\pi}{3} - \sin\left(x - \frac{\pi}{3}\right)\sin\frac{\pi}{3}\\ &= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \cdots\right]\\ &\quad - \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{3}\right) - \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots\right]\\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2}\frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3\\ &\quad + \frac{1}{2}\frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \cdots.\end{aligned}$$

This series representation is valid for all x. A similar calculation would enable us to expand $\cos x$ or $\sin x$ in powers of $x - c$ for any real c; both functions are analytic at every point of the real line. ■

Sometimes it is quite difficult, if not impossible, to find a formula for the general term of a Maclaurin or Taylor series. In such cases it is usually possible to obtain the first few terms before the calculations get too cumbersome. Had we attempted to solve Example 3(c) by multiplying the series for $\sin x$ by itself we might have found ourselves in this bind. Other examples occur when it is necessary to substitute one series into another or to divide one by another.

Example 6 Obtain the first three nonzero terms of the Maclaurin series for (a) $\ln\cos x$, and (b) $\tan x$.

Solution

(a)
$$\begin{aligned}\ln\cos x &= \ln\left(1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\right)\\ &= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2\\ &\quad + \frac{1}{3}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^3 - \cdots\end{aligned}$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots - \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^6}{24} + \cdots\right)$$
$$+ \frac{1}{3}\left(-\frac{x^6}{8} + \cdots\right) - \cdots$$
$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots.$$

Note that at each stage of the calculation we kept only enough terms to ensure that we could get all the terms with powers up to x^6. Being an even function, $\ln\cos x$ has only even powers in its Maclaurin series. We cannot find the general term of this series, and only with considerable computational effort can we find many more terms than we have already found. We could also try to calculate terms by using the formula $a_k = f^{(k)}(0)/k!$ but even this becomes difficult after the first few values of k.

(b) $\tan x = (\sin x)/(\cos x)$. We can obtain the first three terms of the Maclaurin series for $\tan x$ by long division of the series for $\cos x$ into that for $\sin x$:

$$\begin{array}{r|l}
 & x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} & x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \\
 & x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots \\
\cline{2-2}
 & \frac{x^3}{3} - \frac{x^5}{30} + \cdots \\
 & \frac{x^3}{3} - \frac{x^5}{6} + \cdots \\
\cline{2-2}
 & \frac{2x^5}{15} - \cdots \\
 & \frac{2x^5}{15} - \cdots \\
\cline{2-2}
\end{array}$$

Thus $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$.

Again, we cannot easily find all the terms of the series. This Maclaurin series for $\tan x$ converges for $|x| < \pi/2$, but we cannot demonstrate this fact by the techniques we have at our disposal now. Note that the series for $\tan x$ could also have been derived from that of $\ln\cos x$ obtained in part (a) because we have $\tan x = -\frac{d}{dx}\ln\cos x$.

Exercises 9.6

Find Maclaurin series representations for the functions in Exercises 1–14. For what values of x is each representation valid?

1. e^{3x+1}
2. $\cos(2x^3)$
3. $\sin(x - \pi/4)$
4. $\cos(2x - \pi)$
5. $x^2\sin(x/3)$
6. $\cos^2(x/2)$

7. $\sin x \cos x$

8. $\tan^{-1}(5x^2)$

9. $\dfrac{1+x^3}{1+x^2}$

10. $\ln(2+x^2)$

11. $\ln \dfrac{1-x}{1+x}$

12. $(e^{2x^2}-1)/x^2$

13. $\cosh x - \cos x$

14. $\sinh x - \sin x$

Find the required Taylor series representations of the functions in Exercises 15–26. Where is each series representation valid?

15. $f(x) = e^{-2x}$ about the point $x = -1$

16. $f(x) = \sin x$ about the point $x = \pi/2$

17. $f(x) = \cos x$ in powers of $x - \pi$

18. $f(x) = \ln x$ in powers of $x - 3$

19. $f(x) = \ln(2+x)$ in powers of $x - 2$

20. $f(x) = e^{2x+3}$ in powers of $x + 1$

21. $f(x) = \sin x - \cos x$ about $x = \dfrac{\pi}{4}$

22. $f(x) = \cos^2 x$ about $x = \dfrac{\pi}{8}$

23. $f(x) = 1/x^2$ in powers of $x + 2$

24. $f(x) = \dfrac{x}{1+x}$ in powers of $x - 1$

25. $f(x) = x \ln x$ in powers of $x - 1$

26. $f(x) = xe^x$ in powers of $x + 2$

Find the first three nonzero terms in the Maclaurin series for the functions in Exercises 27–30.

27. $\sec x$

28. $\sec x \tan x$

29. $\tan^{-1}(e^x - 1)$

30. $e^{\tan^{-1} x} - 1$

∗31. Use the fact that $(\sqrt{1+x})^2 = 1 + x$ to find the first three nonzero terms of the Maclaurin series for $\sqrt{1+x}$.

32. Does $\csc x$ have a Maclaurin series? Why? Find the first three nonzero terms of the Taylor series for $\csc x$ about the point $x = \pi/2$.

Find the sums of the series in Exercises 33–36.

33. $1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dfrac{x^8}{4!} + \cdots$

∗34. $x^3 - \dfrac{x^9}{3! \times 4} + \dfrac{x^{15}}{5! \times 16} - \dfrac{x^{21}}{7! \times 64} + \dfrac{x^{27}}{9! \times 256} - \cdots$

35. $1 + \dfrac{x^2}{3!} + \dfrac{x^4}{5!} + \dfrac{x^6}{7!} + \dfrac{x^8}{9!} + \cdots$

∗36. $1 + \dfrac{1}{2 \times 2!} + \dfrac{1}{4 \times 3!} + \dfrac{1}{8 \times 4!} + \cdots$

37. Let $P(x) = 1 + x + x^2$. Find (a) the Maclaurin series for $P(x)$ and (b) the Taylor series for $P(x)$ about $x = 1$.

∗38. Verify by direct calculation that $f(x) = 1/x$ is analytic at $x = a$ for every $a \neq 0$.

∗39. Verify by direct calculation that $\ln x$ is analytic at $x = a$ for every $a > 0$.

∗40. Review Exercise 41 of Section 4.3. It shows that the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has derivatives of all orders at every point of the real line, and $f^{(k)}(0) = 0$ for every positive integer k. What is the Maclaurin series for $f(x)$? What is the interval of convergence of this Maclaurin series? On what interval does the series converge to $f(x)$? Is f analytic at $x = 0$?

∗41. By direct multiplication of the Maclaurin series for e^x and e^y show that $e^x e^y = e^{x+y}$.

9.7 Applications of Taylor and Maclaurin Series

Approximating the Values of Functions

We saw in Section 4.8 how Taylor and Maclaurin polynomials (the partial sums of Taylor and Maclaurin series) can be used as polynomial approximations to more complicated functions. In Example 4 of that section we used the Lagrange remainder in Taylor's Formula to determine how many terms of the Maclaurin series for e^x are needed to calculate $e^1 = e$ correct to t3 decimal places. (We will reconsider Taylor's Formula in the next section.) For comparison, we obtained the same result in Example 7 in Section 9.3 by using a geometric series to bound the tail of the series for e.

The following example shows how the error bound associated with the alternating series test (see Theorem 15 in Section 9.4) can also be used for such approximations: when the terms a_n of a series (i) alternate in sign, (ii) decrease steadily in size, and (iii) approach zero as $n \to \infty$, then the error involved in using a partial sum of the series as an approximation to the sum of the series has the same sign as, and is smaller in absolute value than, the first omitted term.

Example 1 Find $\cos 43^\circ$ with error less than $1/10{,}000$.

Solution We give two alternative solutions:

Method I. We can use the Maclaurin series:

$$\cos 43^\circ = \cos\frac{43\pi}{180} = 1 - \frac{1}{2!}\left(\frac{43\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{43\pi}{180}\right)^4 - \cdots.$$

Now $43\pi/180 \approx 0.75049\cdots < 1$, so the series above must satisfy the conditions then the error E will satisfy

$$|E| \le \frac{1}{(2n)!}\left(\frac{43\pi}{180}\right)^{2n} < \frac{1}{(2n)!}.$$

The error will not exceed $1/10{,}000$ if $(2n)! > 10{,}000$, so $n = 4$ will do ($8! = 40{,}320$).

$$\cos 43^\circ \approx 1 - \frac{1}{2!}\left(\frac{43\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{43\pi}{180}\right)^4 - \frac{1}{6!}\left(\frac{43\pi}{180}\right)^6 \approx 0.73135\cdots$$

Method II. Since 43° is close to 45°, we can do a bit better by using the Taylor series about $x = \pi/4$ instead of the Maclaurin series:

$$\begin{aligned}\cos 43^\circ &= \cos\left(\frac{\pi}{4} - \frac{\pi}{90}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{90} + \sin\frac{\pi}{4}\sin\frac{\pi}{90}\\ &= \frac{1}{\sqrt{2}}\left[\left(1 - \frac{1}{2!}\left(\frac{\pi}{90}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{90}\right)^4 - \cdots\right)\right.\\ &\qquad \left. + \left(\frac{\pi}{90} - \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 + \cdots\right)\right].\end{aligned}$$

Since

$$\frac{1}{4!}\left(\frac{\pi}{90}\right)^4 < \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 < \frac{1}{20{,}000},$$

we need only the first two terms of the first series and the first term of the second series:

$$\cos 43^\circ \approx \frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{90} - \frac{1}{2}\left(\frac{\pi}{90}\right)^2\right) \approx 0.731358\cdots.$$

(In fact, $\cos 43^\circ = 0.7313537\cdots$.)

■

When finding approximate values of functions, it is best, whenever possible, to use a power series about a point as close as possible to the point where the approximation is desired.

Functions Defined by Integrals

Many functions that can be expressed as simple combinations of elementary functions cannot be antidifferentiated by elementary techniques; their antiderivatives are not simple combinations of elementary functions. We can, however, often find the Taylor series for the antiderivatives of such functions and hence approximate their definite integrals.

Example 2 Find the Maclaurin series for

$$E(x) = \int_0^x e^{-t^2}\,dt,$$

and use it to evaluate $E(1)$ correct to 3 decimal places.

Solution The Maclaurin series for $E(x)$ is given by

$$\begin{aligned} E(x) &= \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots\right) dt \\ &= \left(t - \frac{t^3}{3} + \frac{t^5}{5 \times 2!} - \frac{t^7}{7 \times 3!} + \frac{t^9}{9 \times 4!} - \cdots\right)\Bigg|_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{5 \times 2!} - \frac{x^7}{7 \times 3!} + \frac{x^9}{9 \times 4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}, \end{aligned}$$

and is valid for all x because the series for e^{-t^2} is valid for all t. Therefore,

$$\begin{aligned} E(1) &= 1 - \frac{1}{3} + \frac{1}{5 \times 2!} - \frac{1}{7 \times 3!} + \cdots \\ &\approx 1 - \frac{1}{3} + \frac{1}{5 \times 2!} - \frac{1}{7 \times 3!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)(n-1)!}. \end{aligned}$$

We stopped with the nth term. The error in this approximation does not exceed the first omitted term, so it will be less than 0.0005, provided $(2n+1)n! > 2{,}000$. Since $13 \times 6! = 9{,}360$, $n = 6$ will do. Thus,

$$E(1) \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1{,}320} \approx 0.747,$$

rounded to three decimal places. ■

Indeterminate Forms

Examples 1 and 2 of Section 4.9 showed how Maclaurin polynomials could be used for evaluating the limits of indeterminate forms. Here are two more examples, this time using the series directly and keeping enough terms to allow cancellation of the [0/0] factors.

Example 3 Evaluate (a) $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$ and (b) $\lim_{x\to 0} \dfrac{(e^{2x} - 1)\ln(1 + x^3)}{(1 - \cos 3x)^2}$.

Solution

(a) $$\lim_{x\to 0} \frac{x - \sin x}{x^3} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{x - \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \cdots}{x^3}$$

$$= \lim_{x\to 0} \left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right) = \frac{1}{3!} = \frac{1}{6}.$$

(b) $$\lim_{x\to 0} \frac{(e^{2x} - 1)\ln(1 + x^3)}{(1 - \cos 3x)^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\left(1 + (2x) + \dfrac{(2x)^2}{2!} + \dfrac{(2x)^3}{3!} + \cdots - 1\right)\left(x^3 - \dfrac{x^6}{2} + \cdots\right)}{\left(1 - \left(1 - \dfrac{(3x)^2}{2!} + \dfrac{(3x)^4}{4!} - \cdots\right)\right)^2}$$

$$= \lim_{x\to 0} \frac{2x^4 + 2x^5 + \cdots}{\left(\dfrac{9}{2}x^2 - \dfrac{3^4}{4!}x^4 + \cdots\right)^2}$$

$$= \lim_{x\to 0} \frac{2 + 2x + \cdots}{\left(\dfrac{9}{2} - \dfrac{3^4}{4!}x^2 + \cdots\right)^2} = \frac{2}{\left(\dfrac{9}{2}\right)^2} = \frac{8}{81}.$$

You can check that the second of these examples is much more difficult if attempted using l'Hôpital's Rule. ■

Exercises 9.7

Use Maclaurin or Taylor series to calculate the function values indicated in Exercises 1–12, with error less than 5×10^{-5} in absolute value.

1. $e^{0.2}$

2. $1/e$

3. $e^{1.2}$

4. $\sin(0.1)$

5. $\cos 5°$

6. $\ln(6/5)$

7. $\ln(0.9)$

8. $\sin 80°$

9. $\cos 65°$

10. $\tan^{-1} 0.2$

11. $\cosh(1)$

12. $\ln(3/2)$

Find Maclaurin series for the functions in Exercises 13–17.

13. $I(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,dt$

14. $J(x) = \displaystyle\int_0^x \frac{e^t - 1}{t}\,dt$

15. $K(x) = \displaystyle\int_1^{1+x} \frac{\ln t}{t - 1}\,dt$

16. $L(x) = \displaystyle\int_0^x \cos(t^2)\,dt$

17. $M(x) = \displaystyle\int_0^x \frac{\tan^{-1} t^2}{t^2}\,dt$

18. Find $L(0.5)$ correct to 3 decimal places, with L defined as in Exercise 16.

19. Find $I(1)$ correct to 3 decimal places, with I defined as in Exercise 13.

Evaluate the limits in Exercises 20–25.

20. $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{\sinh x}$

21. $\displaystyle\lim_{x\to 0} \frac{1 - \cos(x^2)}{(1 - \cos x)^2}$

22. $\lim_{x\to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1 + x^2)}$

23. $\lim_{x\to 0} \frac{2\sin 3x - 3\sin 2x}{5x - \tan^{-1} 5x}$

24. $\lim_{x\to 0} \frac{\sin(\sin x) - x}{x(\cos(\sin x) - 1)}$

25. $\lim_{x\to 0} \frac{\sinh x - \sin x}{\cosh x - \cos x}$

9.8 Taylor's Formula Revisited

Theorem 10 of Section 4.8 (Taylor's Theorem with Lagrange remainder) provides a formula for the error involved when the Taylor polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

of a function $f(x)$ about $x = c$ is used to approximate $f(x)$ for values of $x \neq c$. Specifically, it states the following:

THEOREM 22 **Taylor's Theorem with Lagrange remainder**

If the $(n + 1)$st derivative of f exists on an interval containing c and x, and if $P_n(x)$ is the Taylor polynomial of degree n for f about the point $x = c$, then Taylor's Formula

$$f(x) = P_n(x) + E_n(x)$$

holds, where the error term $E_n(x)$ is given by

$$E_n(x) = \frac{f^{(n+1)}(X)}{(n + 1)!}(x - c)^{n+1}$$

for some X between c and x. ($E_n(x)$ is called the Lagrange remainder in Taylor's Formula.)

Observe that the Lagrange form of the remainder, $E_n(x)$, looks just like the $(n+1)$st degree term in $P_{n+1}(x)$, except that c in $f^{(n+1)}(c)$ has been replaced by an unknown number X between c and x. The cases $n = 0$ and $n = 1$ of Taylor's Formula with Lagrange remainder are just the Mean-Value Theorem (Theorem 11 of Section 2.6) and the error formula for linear approximation (Theorem 9 of Section 4.7), respectively.

Example 1 Use Taylor's Theorem to determine how many terms of the Maclaurin series for $\cos x$ are needed to calculate $\cos 10°$ correctly to 5 decimal places.

Solution Being an even function, $f(x) = \cos x$ has only even degree terms in its Maclaurin series. The Maclaurin polynomials P_{2n} and P_{2n+1} for $f(x)$ are therefore equal:

$$P_{2n}(x) = P_{2n+1}(x) = \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} x^{2j}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n}{(2n)!} x^{2n}.$$

It makes good sense to use the remainder E_{2n+1} rather than the remainder E_{2n}; it is likely to be smaller and therefore assure us of more accuracy for any given value of n. Since $f^{(2n+2)}(x) = (-1)^{n+1} \cos x$, we have, for some X between 0 and x,

$$|E_{2n+1}(x)| = \left| \frac{(-1)^{n+1} \cos X}{(2n+2)!} x^{2n+2} \right| \leq \frac{|x|^{2n+2}}{(2n+2)!}.$$

For $x = 10^\circ = \pi/18 \approx 0.174533 < 0.2$ radians, we will have 5 decimal place accuracy if

$$\frac{0.2^{2n+2}}{(2n+2)!} < 0.000005.$$

This is satisfied if $n = 2$ ($0.2^6/6! < 9 \times 10^{-8}$), but not $n = 1$. Thus,

$$\cos 10^\circ = \cos \frac{\pi}{18} \approx 1 - \frac{1}{2}\left(\frac{\pi}{18}\right)^2 + \frac{1}{24}\left(\frac{\pi}{18}\right)^4 \approx 0.98481$$

to 5 decimal places. ■

Using Taylor's Theorem to Find Taylor and Maclaurin Series

If a function f has derivatives of all orders, then we can write Taylor's Formula for any n:

$$f(x) = P_n(x) + E_n(x).$$

If we can show that $\lim_{n\to\infty} E_n(x) = 0$ for all x in some interval I, then we are entitled to conclude, for x in I, that

$$f(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k,$$

that is, we will have expressed $f(x)$ as the sum of an infinite series of terms which are multiples of positive integer powers of $x - c$, and the series converges for all x in I. This series is the **Taylor series** representation of f in powers of $x - c$ (or the **Maclaurin series** if $c = 0$).

Example 2 Use Taylor's Theorem to find the Maclaurin series for $f(x) = e^x$. Where does the series converge to $f(x)$?

Solution Since e^x is positive and increasing, $e^X \leq e^{|x|}$ for any $X \leq |x|$. Since $f^{(k)}(x) = e^x$ for any k we have, taking $c = 0$ in the Lagrange remainder in Taylor's Formula,

$$|E_n(x)| = \left| \frac{f^{(n+1)}(X)}{(n+1)!} x^{n+1} \right|$$
$$\leq \frac{e^X}{(n+1)!} |x|^{n+1} \leq e^{|x|} \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

for any real x, as shown in Theorem 3(b) of Section 9.1. Therefore,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and the series converges to e^x for all real numbers x. ■

Taylor's Theorem with Integral Remainder

The following theorem is another version of Taylor's Theorem, where the remainder in Taylor's Formula is expressed as an integral.

THEOREM 23 **Taylor's Theorem with integral remainder**

If the $(n+1)$st derivative of f exists on an interval containing c and x, and if $P_n(x)$ is the Taylor polynomial of degree n for f about the point $x = c$, then the remainder $E_n(x) = f(x) - P_n(x)$ in Taylor's Formula is given by

$$E_n(x) = \frac{1}{n!} \int_c^x (x-t)^n f^{(n+1)}(t)\, dt.$$

PROOF We start with the Fundamental Theorem of Calculus written in the form

$$f(x) = f(c) + \int_c^x f'(t)\, dt = P_0(x) + E_0(x).$$

(Note that the Fundamental Theorem is just the special case $n = 0$ of Taylor's Formula with integral remainder.) We now apply integration by parts to the integral, setting

$$U = f'(t), \qquad dV = dt,$$
$$dU = f''(t)\, dt, \qquad V = -(x-t).$$

(We have broken our usual rule about not including a constant of integration with V. In this case we have included the constant $-x$ in V in order to have V vanish when $t = x$.) We have

$$f(x) = f(c) - f'(t)(x-t)\Big|_{t=c}^{t=x} + \int_c^x (x-t) f''(t)\, dt$$
$$= f(c) + f'(c)(x-c) + \int_c^x (x-t) f''(t)\, dt$$
$$= P_1(x) + E_1(x).$$

We have thus proved the case $n = 1$ of Taylor's Formula with integral remainder.

Let us complete the proof for general n by mathematical induction. Suppose that Taylor's Formula holds with integral remainder for some $n = k$:

$$f(x) = P_k(x) + E_k(x) = P_k(x) + \frac{1}{k!} \int_c^x (x-t)^k f^{(k+1)}(t)\, dt.$$

Again we integrate by parts. Let

$$U = f^{(k+1)}(t), \qquad dV = (x-t)^k\,dt,$$
$$dU = f^{(k+2)}(t)\,dt, \qquad V = \frac{-1}{k+1}(x-t)^{k+1}.$$

We have

$$\begin{aligned} f(x) &= P_k(x) + \frac{1}{k!}\left(-\frac{f^{(k+1)}(t)(x-t)^{k+1}}{k+1}\bigg|_{t=c}^{t=x} + \int_c^x \frac{(x-t)^{k+1} f^{(k+2)}(t)}{k+1}\,dt\right) \\ &= P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!}(x-c)^{k+1} + \frac{1}{(k+1)!}\int_c^x (x-t)^{k+1} f^{(k+2)}(t)\,dt \\ &= P_{k+1}(x) + E_{k+1}(x). \end{aligned}$$

Thus Taylor's Formula with integral remainder is valid for $n = k+1$ if it is valid for $n = k$. Having been shown to be valid for $n = 0$ (and $n = 1$), it must therefore be valid for every positive integer n for which $E_n(x)$ exists.

Remark Using one or the other of the versions of Taylor's Theorem given in this section, all the basic Maclaurin and Taylor series given in Section 9.6 can be verified without having to use the theory of power series.

Exercises 9.8

1. Estimate the error if the Maclaurin polynomial of degree 5 for $\sin x$ is used to approximate $\sin(0.2)$.
2. Estimate the error if the Maclaurin polynomial of degree 6 for $\cos x$ is used to approximate $\cos(1)$.
3. Estimate the error if the Maclaurin polynomial of degree 4 for e^{-x} is used to approximate $e^{-0.5}$.
4. Estimate the error if the Maclaurin polynomial of degree 2 for $\sec x$ is used to approximate $\sec(0.2)$.
5. Estimate the error if the Maclaurin polynomial of degree 3 for $\ln(\cos x)$ is used to approximate $\ln(\cos 0.1)$.
6. Estimate the error if the Taylor polynomial of degree 3 for $\tan^{-1} x$ in powers of $x - 1$ is used to approximate $\tan^{-1} 0.99$.
7. Estimate the error if the Taylor polynomial of degree 4 for $\ln x$ in powers of $x - 2$ is used to approximate $\ln(1.95)$.

Use Taylor's Formula to establish the Maclaurin series for the functions in Exercises 8–15.

8. e^{-x}
9. 2^x
10. $\cos x$
11. $\sin x$
12. $\sin^2 x$
13. $\dfrac{1}{1-x}$

* 14. $\ln(1+x)$ (Use the integral remainder.)

* 15. $\dfrac{x}{2+3x}$ (Use Exercise 13.)

Use Taylor's Formula to obtain the Taylor series indicated in Exercises 16–21.

16. for e^x in powers of $x - a$
17. for $\sin x$ in powers of $x - (\pi/6)$
18. for $\cos x$ in powers of $x - (\pi/4)$

* 19. for $\ln x$ in powers of $x - 1$ (Use the integral remainder.)

* 20. for $\ln x$ in powers of $x - 2$

21. for $1/x$ in powers of $x + 2$ (Use Exercise 13.)

9.9 The Binomial Theorem and Binomial Series

Example 1 Use Taylor's Formula to prove the Binomial Theorem: if n is a positive integer, then

$$(a+x)^n = a^n + n\,a^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots + n\,ax^{n-1} + x^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}x^k,$$

where $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$.

Solution Let $f(x) = (a+x)^n$. Then

$$f'(x) = n(a+x)^{n-1} = \frac{n!}{(n-1)!}(a+x)^{n-1}$$
$$f''(x) = \frac{n!}{(n-1)!}(n-1)(a+x)^{n-2} = \frac{n!}{(n-2)!}(a+x)^{n-2}$$
$$\vdots$$
$$f^{(k)}(x) = \frac{n!}{(n-k)!}(a+x)^{n-k} \qquad (0 \le k \le n).$$

In particular, $f^{(n)}(x) = \dfrac{n!}{0!}(a+x)^{n-n} = n!$, a constant, and

$$f^{(k)}(x) = 0 \qquad \text{for all } x, \text{ if } k > n.$$

For $0 \le k \le n$ we have $f^{(k)}(0) = \dfrac{n!}{(n-k)!}a^{n-k}$. Thus, by Taylor's Theorem with Lagrange remainder,

$$(a+x)^n = f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(X)}{(n+1)!}x^{n+1}$$
$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}a^{n-k}x^k + 0 = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}x^k.$$

This is, in fact, the Maclaurin *series* for $(a+x)^n$, not just the Maclaurin polynomial of degree n. Since all higher-degree terms are zero, the series has only finitely many nonzero terms and so converges for all x. ■

Remark If $f(x) = (a+x)^r$, where $a > 0$ and r is any real number, then calculations similar to those above show that the Maclaurin polynomial of degree n for f is

$$P_n(x) = a^r + \sum_{k=1}^{n} \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}a^{r-k}x^k.$$

However, if r is not a positive integer, then there will be no positive integer n for which the remainder $E_n(x) = f(x) - P_n(x)$ vanishes identically, and the corresponding Maclaurin series will not be a polynomial.

The Binomial Series

To simplify the discussion of the function $(a+x)^r$ when r is not a positive integer, we take $a = 1$ and consider the function $(1+x)^r$. Results for the general case follow via the identity

$$(a+x)^r = a^r\left(1+\frac{x}{a}\right)^r,$$

valid for any $a > 0$.

If r is any real number and $x > -1$, then the kth derivative of $(1+x)^r$ is

$$r(r-1)(r-2)\cdots(r-k+1)\,(1+x)^{r-k}, \qquad (k = 1, 2, \ldots).$$

Thus, the Maclaurin series for $(1+x)^r$ is

$$1+\sum_{k=1}^{\infty}\frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}x^k,$$

which is called the **binomial series**. The following theorem shows that the binomial series does, in fact, converge to $(1+x)^r$ if $|x| < 1$. We could accomplish this by writing Taylor's Formula for $(1+x)^r$ with $c = 0$ and showing that the remainder $E_n(x) \to 0$ as $n \to \infty$. (We would need to use the integral form of the remainder to prove this for all $|x| < 1$.) However, we will use an easier method, similar to the one used for the exponential and trigonometric functions in Section 9.6.

THEOREM 24 The binomial series

If $|x| < 1$, then

$$\begin{aligned}(1+x)^r &= 1+rx+\frac{r(r-1)}{2!}x^2+\frac{r(r-1)(r-2)}{3!}x^3+\cdots\\ &= 1+\sum_{n=1}^{\infty}\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n \quad (-1 < x < 1).\end{aligned}$$

PROOF If $|x| < 1$, then the series

$$f(x) = 1+\sum_{n=1}^{\infty}\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n$$

converges by the ratio test, since

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{\dfrac{r(r-1)(r-2)\cdots(r-n+1)(r-n)}{(n+1)!}x^{n+1}}{\dfrac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n}\right|\\ &= \lim_{n\to\infty}\left|\frac{r-n}{n+1}\right||x| = |x| < 1.\end{aligned}$$

Note that $f(0) = 1$. We need to show that $f(x) = (1+x)^r$ for $|x| < 1$.

By Theorem 19, we can differentiate the series for $f(x)$ termwise on $|x| < 1$ to obtain

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{(n-1)!} x^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n)}{n!} x^{n}. \end{aligned}$$

We have replaced n with $n+1$ to get the second version of the sum from the first version. Adding the second version to x times the first version, we get

$$\begin{aligned} (1+x)f'(x) &= \sum_{n=0}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n)}{n!} x^{n} \\ &\quad + \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{(n-1)!} x^{n} \\ &= r + \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!} x^{n} \left[(r-n)+n\right] \\ &= r\, f(x). \end{aligned}$$

The differential equation $(1+x)f'(x) = rf(x)$ implies that

$$\frac{d}{dx}\frac{f(x)}{(1+x)^r} = \frac{(1+x)^r f'(x) - r(1+x)^{r-1} f(x)}{(1+x)^{2r}} = 0$$

for all x satisfying $|x| < 1$. Thus, $f(x)/(1+x)^r$ is constant on that interval, and since $f(0) = 1$, the constant must be 1. Thus $f(x) = (1+x)^r$.

Remark For some values of r the binomial series may converge at the endpoints $x = 1$ or $x = -1$. As observed above, if r is a positive integer, the series has only finitely many nonzero terms, and so converges for all x.

Example 2 Find the Maclaurin series for $\dfrac{1}{\sqrt{1+x}}$.

Solution Here $r = -(1/2)$:

$$\begin{aligned} \frac{1}{\sqrt{1+x}} &= (1+x)^{-1/2} \\ &= 1 - \frac{1}{2}x + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2 + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{1\times 3}{2^2 2!}x^2 - \frac{1\times 3\times 5}{2^3 3!}x^3 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n n!} x^n. \end{aligned}$$

This series converges for $-1 < x \le 1$. (Use the alternating series test to get the endpoint $x = 1$.)

Example 3 Find the Maclaurin series for $\sin^{-1} x$.

Solution Replace x with $-t^2$ in the series obtained in the previous example to get

$$\frac{1}{\sqrt{1-t^2}} = 1 + \sum_{n=1}^{\infty} \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n n!} t^{2n} \qquad (-1 < t < 1).$$

Now integrate t from 0 to x:

$$\begin{aligned}\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n n!} t^{2n}\right) dt \\ &= x + \sum_{n=1}^{\infty} \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n n!(2n+1)} x^{2n+1} \\ &= x + \frac{x^3}{6} + \frac{3}{40} x^5 + \cdots \qquad (-1 < x < 1).\end{aligned}$$

Exercises 9.9

Find Maclaurin series representations for the functions in Exercises 1–6. Use the binomial series to calculate the answers.

1. $\sqrt{1+x}$
2. $x\sqrt{1-x}$
3. $\sqrt{4+x}$
4. $\dfrac{1}{\sqrt{4+x^2}}$
5. $(1-x)^{-2}$
6. $(1+x)^{-3}$

* 7. **(Binomial coefficients)** Show that the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

satisfy

(i) $\binom{n}{0} = \binom{n}{n} = 1$ for every n, and

(ii) if $0 \le k \le n$, then $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

It follows that, for fixed $n \ge 1$, the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ are the elements of the nth row of **Pascal's triangle**:

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

where each element with value > 1 is the sum of the two diagonally above it.

* 8. **(An inductive proof of the Binomial Theorem)** Use mathematical induction and the results of Exercise 7 to prove the Binomial Theorem:

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + na^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \cdots + b^n.\end{aligned}$$

* 9. **(The Leibniz Rule)** Use mathematical induction, the Product Rule, and Exercise 7 to verify the Leibniz Rule for the nth derivative of a product of two functions:

$$\begin{aligned}(fg)^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \\ &= f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2} f^{(n-2)}g'' \\ &\quad + \binom{n}{3} f^{(n-3)}g^{(3)} + \cdots + fg^{(n)}.\end{aligned}$$

9.10 Series Solutions of Differential Equations

In Section 3.7 we developed a recipe for solving second-order, linear, homogeneous differential equations with constant coefficients:

$$ay'' + by' + cy = 0.$$

Many of the second-order, linear, homogeneous differential equations that arise in applications do not have constant coefficients. If the coefficient functions of such an equation are sufficiently well behaved, we can often find solutions in the form of power series (Taylor series). Such series solutions are frequently used to define new functions, whose properties are deduced partly from the fact that they solve particular differential equations. For example, Bessel functions of order ν are defined to be certain series solutions of Bessel's differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

Series solutions for second-order homogeneous linear differential equations are most easily found near an **ordinary point** of the equation. This is a point $x = a$ such that the equation can be expressed in the form

$$y'' + p(x)y' + q(x)y = 0,$$

where the functions $p(x)$ and $q(x)$ are **analytic** at $x = a$. (Recall that a function f is analytic at $x = a$ if $f(x)$ can be expressed as the sum of its Taylor series in powers of $x - a$ in an interval of positive radius centred at $x = a$.) Thus we assume

$$p(x) = \sum_{n=0}^{\infty} p_n(x-a)^n,$$
$$q(x) = \sum_{n=0}^{\infty} q_n(x-a)^n,$$

with both series converging in some interval of the form $a - R < x < a + R$. Frequently $p(x)$ and $q(x)$ are polynomials, so are analytic everywhere. A change of independent variable $\xi = x - a$ will put the point $x = a$ at the origin $\xi = 0$, so we can assume that $a = 0$.

The following example illustrates the technique of series solution around an ordinary point.

Example 1 Find two independent solutions in powers of x for the Hermite equation

$$y'' - 2xy' + \nu y = 0.$$

For what values of ν does the equation have a polynomial solution?

Solution We try for a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \qquad \text{so that}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

(We have replaced n by $n+2$ in order to get x^n in the sum for y''.) We substitute these expressions into the differential equation to get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=1}^{\infty} n a_n x^n + \nu \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\text{or} \quad 2a_2 + \nu a_0 + \sum_{n=1}^{\infty} \Big[(n+2)(n+1) a_{n+2} - (2n - \nu) a_n \Big] x^n = 0.$$

This identity holds for all x provided that the coefficient of every power of x vanishes; that is,

$$a_2 = -\frac{\nu a_0}{2}, \qquad a_{n+2} = \frac{(2n-\nu) a_n}{(n+2)(n+1)}, \qquad (n = 1, 2, \cdots).$$

The latter of these formulas is called a **recurrence relation**.

We can choose a_0 and a_1 to have any values; then the above conditions determine all the remaining coefficients a_n, $(n \geq 2)$. We can get one solution by choosing, for instance, $a_0 = 1$ and $a_1 = 0$. Then, by the recurrence relation,

$$a_3 = 0, \quad a_5 = 0, \quad a_7 = 0, \quad \cdots, \qquad \text{and}$$

$$a_2 = -\frac{\nu}{2}$$

$$a_4 = \frac{(4-\nu) a_2}{4 \times 3} = -\frac{\nu(4-\nu)}{2 \times 3 \times 4} = -\frac{\nu(4-\nu)}{4!}$$

$$a_6 = \frac{(8-\nu) a_4}{6 \times 5} = -\frac{\nu(4-\nu)(8-\nu)}{6!}$$

$$\cdots$$

The pattern is obvious here:

$$a_{2n} = -\frac{\nu(4-\nu)(8-\nu)\cdots(4n-4-\nu)}{(2n)!}, \qquad (n = 1, 2, \cdots).$$

One solution to the Hermite equation is

therefore also vanish. [illegible]

The second solution, y_2, can be found in the same way, by choosing $a_0 = 0$ and $a_1 = 1$. It is

$$y_2 = x + \sum_{n=1}^{\infty} \frac{(2-\nu)(6-\nu)\cdots(4n-2-\nu)}{(2n+1)!} x^{2n+1},$$

and it is an odd polynomial of degree $2n+1$ if $\nu = 4n+2$.

Both of these series solutions converge for all x. The ratio test can be applied directly to the recurrence relation. Since consecutive nonzero terms of each series are of the form $a_n x^n$ and $a_{n+2}x^{n+2}$, we calculate

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+2}x^{n+2}}{a_n x^n} \right| = |x|^2 \lim_{n\to\infty} \left| \frac{a_{n+2}}{a_n} \right| = |x|^2 \lim_{n\to\infty} \left| \frac{2n-\nu}{(n+2)(n+1)} \right| = 0$$

for every x, so the series converges by the ratio test. ■

If $x = a$ is not an ordinary point of the equation

$$y'' + p(x)y' + q(x)y = 0,$$

then it is called a **singular point** of that equation. This means that at least one of the functions $p(x)$ and $q(x)$ is not analytic at $x = a$. If, however, $(x-a)p(x)$ and $(x-a)^2 q(x)$ are analytic at $x = a$, then the singular point is said to be a **regular singular point**. For example, the origin $x = 0$ is a regular singular point of Bessel's equation,

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0,$$

since $p(x) = 1/x$ and $q(x) = (x^2 - \nu^2)/x^2$ satisfy $xp(x) = 1$ and $x^2 q(x) = x^2 - \nu^2$, which are both polynomials and therefore analytic.

The solutions of differential equations are usually not analytic at singular points. However, it is still possible to find at least one series solution about such a point. The method involves searching for a series solution of the form x^μ times a power series, that is,

$$y = (x-a)^\mu \sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} a_n (x-a)^{n+\mu}, \qquad \text{where } a_0 \neq 0.$$

Substitution into the differential equation produces a quadratic **indicial equation**, which determines one or two values of μ for which such solutions can be found, and a **recurrence relation** enabling the coefficients a_n to be calculated for $n \geq 1$. If the indicial roots are not equal and do not differ by an integer, two independent solutions can be calculated. If the indicial roots are equal or differ by an integer, one such solution can be calculated (corresponding to the larger indicial root), but finding a second independent solution (and so the general solution) requires techniques beyond the scope of this book. The reader is referred to standard texts on differential equations for more discussion and examples. We will content ourselves here with one final example.

Example 2 Find one solution, in powers of x, of Bessel's equation of order $\nu = 1$, namely,

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

Solution We try

$$y = \sum_{n=0}^{\infty} a_n x^{\mu+n}$$

$$y' = \sum_{n=0}^{\infty} (\mu + n)a_n x^{\mu+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (\mu + n)(\mu + n - 1)a_n x^{\mu+n-2}.$$

Substituting these expressions into the Bessel equation, we get

$$\sum_{n=0}^{\infty}\Big[\big((\mu + n)(\mu + n - 1) + (\mu + n) - 1\big)a_n x^n + a_n x^{n+2}\Big] = 0$$

$$\sum_{n=0}^{\infty}\Big[(\mu + n)^2 - 1\Big]a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$(\mu^2 - 1)a_0 + \big((\mu + 1)^2 - 1\big)a_1 x + \sum_{n=2}^{\infty}\Big[\big((\mu + n)^2 - 1\big)a_n + a_{n-2}\Big]x^n = 0.$$

All of the terms must vanish. Since $a_0 \neq 0$ (we may take $a_0 = 1$) we obtain

$$\mu^2 - 1 = 0, \qquad \text{the indicial equation}$$

$$[(\mu + 1)^2 - 1]a_1 = 0,$$

$$a_n = -\frac{a_{n-2}}{(\mu + n)^2 - 1}, \quad (n \geq 2). \qquad \text{the recurrence relation}$$

Evidently $\mu = \pm 1$; therefore $a_1 = 0$. If we take $\mu = 1$, then the recurrence relation is $a_n = -a_{n-2}/(n)(n + 2)$. Thus,

$$a_3 = 0, \quad a_5 = 0, \quad a_7 = 0, \quad \cdots$$

$$a_2 = \frac{-1}{2 \times 4}, \quad a_4 = \frac{1}{2 \times 4 \times 4 \times 6}, \quad a_6 = \frac{-1}{2 \times 4 \times 4 \times 6 \times 6 \times 8}, \quad \cdots.$$

Again the pattern is obvious:

$$a_{2n} = \frac{(-1)^n}{2^{2n}n!(n + 1)!},$$

and one solution of the Bessel equation of order 1 is

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}n!(n + 1)!} x^{2n+1}.$$

By the ratio test, this series converges for all x. ■

Remark Observe that if we tried to calculate a second solution using $\mu = -1$ we would get the recurrence relation

$$a_n = -\frac{a_{n-2}}{n(n-2)},$$

and we would be unable to calculate a_2. This shows what can happen if the indicial roots differ by an integer.

Exercises 9.10

1. Find the general solution of $y'' = (x-1)^2 y$ in the form of a power series $y = \sum_{n=0}^{\infty} a_n (x-1)^n$.
2. Find the general solution of $y'' = xy$ in the form of a power series $y = \sum_{n=0}^{\infty} a_n x^n$ with a_0 and a_1 arbitrary.
3. Solve the initial-value problem

$$\begin{cases} y'' + xy' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 2. \end{cases}$$

4. Find the solution of $y'' + xy' + y = 0$ that satisfies $y(0) = 1$ and $y'(0) = 0$.
5. Find the first three nonzero terms in a power series solution in powers of x for the initial-value problem $y'' + (\sin x)y = 0$, $y(0) = 1$, $y'(0) = 0$.
6. Find the solution, in powers of x, for the initial-value problem

$$(1-x^2)y'' - xy' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

7. Find two power series solutions in powers of x for $3xy'' + 2y' + y = 0$.
8. Find one power series solution for the Bessel equation of order $\nu = 0$, that is, the equation $xy'' + y' + xy = 0$.

Chapter Review

Key Ideas

- **What does it mean to say that the sequence $\{a_n\}$**
  - ◇ is bounded above?
  - ◇ is ultimately positive?
  - ◇ is alternating?
  - ◇ is increasing?
  - ◇ converges?
  - ◇ diverges to infinity?
- **What does it mean to say that the series $\sum_{n=1}^{\infty} a_n$**
  - ◇ converges?
  - ◇ diverges?
  - ◇ is geometric?
  - ◇ is telescoping?
  - ◇ is a p-series?
  - ◇ is positive?
  - ◇ converges absolutely?
  - ◇ converges conditionally?
- **State the following convergence tests for series.**
  - ◇ the integral test
  - ◇ the comparison test
  - ◇ the limit comparison test
  - ◇ the ratio test
  - ◇ the alternating series test
- **How can you find bounds for the tail of a series?**
- **What is a bound for the tail of an alternating series?**
- **What do the following terms and phrases mean?**
  - ◇ a power series
  - ◇ interval of convergence
  - ◇ radius of convergence
  - ◇ centre of convergence
  - ◇ a Taylor series
  - ◇ a Maclaurin series
  - ◇ a Taylor polynomial
  - ◇ a binomial series
  - ◇ an analytic function
- **Where is the sum of a power series differentiable?**
- **Where does the integral of a power series converge?**
- **Where is the sum of a power series continuous?**
- **State Taylor's Theorem with Lagrange remainder.**
- **State Taylor's Theorem with integral remainder.**
- **What is the binomial theorem?**

Review Exercises

In Exercises 1–4, determine whether the given sequence does or does not converge, and find its limit if it does converge.

1. $\left\{ \dfrac{(-1)^n e^n}{n!} \right\}$
2. $\left\{ \dfrac{n^{100} + 2^n \pi}{2^n} \right\}$
3. $\left\{ \dfrac{\ln n}{\tan^{-1} n} \right\}$
4. $\left\{ \dfrac{(-1)^n n^2}{\pi n (n - \pi)} \right\}$

5. Let $a_1 > \sqrt{2}$, and let

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Show that $\{a_n\}$ is decreasing and that $a_n > \sqrt{2}$ for $n \geq 1$. Why must $\{a_n\}$ converge? Find $\lim_{n\to\infty} a_n$.

6. Find the limit of the sequence $\{\ln\ln(n+1) - \ln\ln n\}$.

Evaluate the sums of the series in Exercises 7–10.

7. $\displaystyle\sum_{n=1}^{\infty} 2^{-(n-5)/2}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n-1}}{(\pi-1)^{2n}}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{1}{4}}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{9}{4}}$

Determine whether the series in Exercises 11–16 converge or diverge. Give reasons for your answers.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{1+3^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(1+n)(1+n\sqrt{n})}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(1+2^n)(1+n\sqrt{n})}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n+1}}{n!}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!+1}$

Do the series in Exercises 17–20 converge absolutely, converge conditionally, or diverge?

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1+n^3}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n - n}$

19. $\displaystyle\sum_{n=10}^{\infty} \frac{(-1)^{n-1}}{\ln\ln n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\cos(n\pi)}{1+n^3}$

For what values of x do the series in Exercises 21–22 converge absolutely? converge conditionally? diverge?

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n\sqrt{n}}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(5-2x)^n}{n}$

Determine the sums of the series in Exercises 23–24 to within 0.001.

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+n^2}$

In Exercises 25–32, find Maclaurin series for the given functions. State where each series converges to the function.

25. $\dfrac{1}{3-x}$

26. $\dfrac{x}{3-x^2}$

27. $\ln(e+x^2)$

28. $\dfrac{1-e^{-2x}}{x}$

29. $x\cos^2 x$

30. $\sin(x+(\pi/3))$

31. $(8+x)^{-1/3}$

32. $(1+x)^{1/3}$

Find Taylor series for the functions in Exercises 33–34 about the indicated points $x = c$.

33. $1/x, \quad c = \pi$

34. $\sin x + \cos x, \quad c = \pi/4$

Find the Maclaurin polynomial of the indicated degree for the functions in Exercises 35–38.

35. e^{x^2+2x}, degree 3

36. $\sin(1+x)$, degree 3

37. $\cos(\sin x)$, degree 4

38. $\sqrt{1+\sin x}$, degree 4

39. What function has Maclaurin series

$$1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}?$$

40. A function $f(x)$ has Maclaurin series

$$1 + x^2 + \frac{x^4}{2^2} + \frac{x^6}{3^2} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n^2}.$$

Find $f^{(k)}(0)$ for all positive integers k.

Find the sums of the series in 41–44.

41. $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{\pi^n}$

* 42. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{\pi^n}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{ne^n}$

* 44. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n\pi^{2n-4}}{(2n-1)!}$

45. If $S(x) = \displaystyle\int_0^x \sin(t^2)\,dt$, find $\displaystyle\lim_{x\to 0} \frac{x^3 - 3S(x)}{x^7}$.

46. Use series to evaluate $\displaystyle\lim_{x\to 0} \frac{(x - \tan^{-1}x)(e^{2x}-1)}{2x^2 - 1 + \cos(2x)}$.

47. How many nonzero terms in the Maclaurin series for e^{-x^4} are needed to evaluate $\int_0^{1/2} e^{-x^4}\,dx$ correct to 5 decimal places? Evaluate the integral to that accuracy.

48. Estimate the size of the error if the Taylor polynomial of degree 4 about $x = \pi/2$ for $f(x) = \ln\sin x$ is used to approximate $\ln\sin(1.5)$.

Challenging Problems

1. **(A refinement of the ratio test)** Suppose $a_n > 0$ and $a_{n+1}/a_n \geq n/n+1$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges. *Hint:* $a_n \geq K/n$ for some constant K.

* 2. **(Summation by parts)** Let $\{u_n\}$ and $\{v_n\}$ be two sequences, and let $s_n = \sum_{k=1}^{n} v_k$.

(a) Show that $\sum_{k=1}^{n} u_k v_k = u_{n+1}s_n + \sum_{k=1}^{n}(u_k - u_{k+1})s_n$. (*Hint:* write $v_n = s_n - s_{n-1}$, with $s_0 = 0$, and rearrange the sum.)

(b) If $\{u_n\}$ is positive, decreasing, and convergent to 0, and if $\{v_n\}$ has bounded partial sums, $|s_n| \leq K$ for all n, where K is a constant, show that $\sum_{n=1}^{\infty} u_n v_n$ converges. (*Hint:* show that the series $\sum_{n=1}^{\infty}(u_n - u_{n+1})s_n$ converges by comparing it to the telescoping series $\sum_{n=1}^{\infty}(u_n - u_{n+1})$.)

* 3. Show that $\sum_{n=1}^{\infty}(1/n)\sin(nx)$ converges for every x. *Hint:* if x is an integer multiple of π, all the terms in the series are 0 so there is nothing to prove. Otherwise, $\sin(x/2) \neq 0$. In this case show that

$$\sum_{n=1}^{N} \sin(nx) = \frac{\cos(x/2) - \cos((N+1/2)x)}{2\sin(x/2)}$$

using the identity

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}$$

to make the sum telescope. Then apply the result of Exercise 2(b) with $u_n = 1/n$ and $v_n = \sin(nx)$.

4. Let $a_1, a_2, a_3, \ldots$ be those positive integers that do not contain the digit 0 in their decimal representations. Thus $a_1 = 1$, $a_2 = 2, \ldots a_9 = 9$, $a_{10} = 11, \ldots a_{18} = 19$, $a_{19} = 21$, $\ldots a_{90} = 99$, $a_{91} = 111$, etc. Show that the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges and that the sum is less than 90. (*Hint:* How many of these integers have m digits? Each term $1/a_n$, where a_n has m digits, is less than 10^{-m+1}.)

* 5. **(Using an integral to improve convergence)** Recall the error formula for the Midpoint Rule, according to which

$$\int_{k-1/2}^{k+1/2} f(x)\,dx - f(k) = \frac{f''(c)}{24},$$

where $k - (1/2) \leq c \leq k + (1/2)$.

(a) If $f''(x)$ is a decreasing function of x, show that

$$f'(k+\tfrac{3}{2}) - f'(k+\tfrac{1}{2}) \leq f''(c) \leq f'(k-\tfrac{1}{2}) - f'(k-\tfrac{3}{2}).$$

(b) If (i) $f''(x)$ is a decreasing function of x, (ii) $\int_{N+1/2}^{\infty} f(x)\,dx$ converges, and (iii) $f'(x) \to 0$ as $x \to \infty$, show that

$$\frac{f'(N-\frac{1}{2})}{24} \leq \sum_{n=N+1}^{\infty} f(n) - \int_{N+1/2}^{\infty} f(x)\,dx \leq \frac{f'(N+\frac{3}{2})}{24}.$$

(c) Use the result of part (b) to approximate $\sum_{n=1}^{\infty} 1/n^2$ to within 0.001.

* 6. **(The number e is irrational.)** Start with $e = \sum_{n=0}^{\infty} 1/n!$.

(a) Use the technique of Example 7 in Section 9.3 to show that for any $n > 0$,

$$0 < e - \sum_{j=0}^{n} \frac{1}{j!} < \frac{1}{n!n}.$$

(Note that the sum here has $n+1$ terms rather than n terms.)

(b) Suppose that e is a rational number, say $e = M/N$ for certain positive integers M and N. Show that $N!\left(e - \sum_{j=0}^{N}(1/j!)\right)$ is an integer.

(c) Combine parts (a) and (b) to show that there is an integer between 0 and $1/N$. Why is this not possible? Conclude that e cannot be a rational number.

7. Let

$$f(x) = \sum_{k=0}^{\infty} \frac{2^{2k}k!}{(2k+1)!} x^{2k+1}$$
$$= x + \frac{2}{3}x^3 + \frac{4}{3\times 5}x^5 + \frac{8}{3\times 5\times 7}x^7 + \ldots.$$

(a) Find the radius of convergence of this power series.

(b) Show that $f'(x) = 1 + 2xf(x)$.

(c) What is $\frac{d}{dx}\left(e^{-x^2} f(x)\right)$?

(d) Express $f(x)$ in terms of an integral.

* 8. **(The number π is irrational)** Problem 6 above shows how to prove that e is irrational by assuming the contrary and deducing a contradiction. In this problem you will show that π is also irrational. The proof for π is also by contradiction but is rather more complicated, so it will be broken down into several parts.

(a) Let $f(x)$ be a polynomial, and let

$$g(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots$$
$$= \sum_{j=0}^{\infty} (-1)^j f^{(2j)}(x).$$

(Since f is a polynomial, all but a finite number of terms in the above sum are identically zero, so there are no convergence problems.) Verify that

$$\frac{d}{dx}\left(g'(x)\sin x - g(x)\cos x\right) = f(x)\sin x,$$

and hence that

$$\int_0^{\pi} f(x)\sin x\,dx = g(\pi) + g(0).$$

(b) Suppose that π is rational, say $\pi = m/n$, where m and n are positive integers. You will show that this leads to a contradiction and thus cannot be true. Choose a positive integer k such that $(\pi m)^k/k! < 1/2$. (Why is this possible?) Consider the polynomial

$$f(x) = \frac{x^k(m-nx)^k}{k!} = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j} m^{k-j}(-n)^j x^{j+k}.$$

Show that $0 < f(x) < 1/2$ for $0 < x < \pi$, and hence that

$$0 < \int_0^{\pi} f(x) \sin x \, dx < 1.$$

Thus, $0 < g(\pi) + g(0) < 1$, where $g(x)$ is defined as in part (a).

(c) Show that the ith derivative of $f(x)$ is given by

$$f^{(i)}(x) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} m^{k-j} (-n)^j \frac{(j+k)!}{(j+k-i)!} x^{j+k-i}.$$

(d) Show that $f^{(i)}(0)$ is an integer for $i = 0, 1, 2, \ldots$. (*Hint:* Observe for $i < k$ that $f^{(i)}(0) = 0$, and for $i > 2k$ that $f^{(i)}(x) = 0$ for all x. For $k \le i \le 2k$, show that only one term in the sum for $f^{(i)}(0)$ is not 0, and that this term is an integer. You will need the fact that the binomial coefficients $\binom{k}{j}$ are integers.)

(e) Show that $f(\pi - x) = f(x)$ for all x, and hence that $f^{(i)}(\pi)$ is also an integer for each $i = 0, 1, 2, \ldots$. Therefore, if $g(x)$ is defined as in (a), then $g(\pi) + g(0)$ is an integer. This contradicts the conclusion of part (b) and so shows that π cannot be rational.

* **9. (An asymptotic series)** Use integration by parts to show that

$$\int_0^x e^{-1/t} \, dt = e^{-1/x} \sum_{n=2}^{N} (-1)^n (n-1)! x^n + (-1)^{N+1} N! \int_0^x t^{N-1} e^{-1/t} \, dt.$$

Why can't you just use a Maclaurin series to approximate this integral? Using $N = 5$, find an approximate value for $\int_0^{0.1} e^{-1/t} \, dt$, and estimate the error. Estimate the error for $N = 10$ and $N = 20$.

Note that the series $\sum_{n=2}^{\infty} (-1)^n (n-1)! x^n$ **diverges** for any $x \neq 0$. This is an example of what is called an **asymptotic series**. Even though it diverges, a properly chosen partial sum gives a good approximation to our function when x is small.